CO-ORDINATE & SOLID GEOMETRY

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PREFACE

This book is written strictly according to the syllabus prescribed by the Calcutta and Burdwan Universities for Pre-University and University Entrance Examinations. Many examples have been worked out and important formulæ have been given at the beginning for ready reference. In order to acquaint the students with the standard of examination some University questions of recent years have been given in the end. It is hoped that the book will meet the requirements of those for whom it is intended.

The book has been thoroughly revised and many examples have been added in the exercises. Any error or omission brought to our notice or any suggestion for further improvement will be gratefully received.

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SYLLABUS

(A) CO-ORDINATE PLANE GEOMETRY

Plane Cartesian co-ordinates, distance between two points, co-ordinates of the point dividing a finite straight line in a given ratio. Area of a triangle.

Equation of a locus in rectangular Cartesian co-ordinates. Transfer of origin without rotation of axes. Equations of a straight line in different forms. Angle between two straight lines, conditions for parallelism and perpendicularity. Perpendicular distance of a point from a given line. Equations of the angle bisectors between two lines. Equation of a circle. Intersection of a straight line and a circle. Condition of tangency. Equation of the tangent and normal at a point.

Definition of a conic with reference to focus and directrix ; a parabola, an ellipse and a hyperbola. Deduction, from difination of the equations of the above loci referred to the directrix and the perpendicular from the focus upon the directrix as axes. Reduction of these equations to their standard forms. Intersection of a straight line with any of the above loci ; condition for tangency. Equation of tangent and normal at a point for each of the above loci. Deduction of simple properties of the above loci.

(B) SOLID GEOMETRY

Definitions—Parallel and skew straight lines. Angle between two straight lines and between two planes, parallelism and perpendicularity. Angle between a plane and a straight line, their parallelism and perpendicularity. Projection of a line on a plane.

Axioms—(i) One and only one plane passes through a given line and a given point outside it. (ii) If two planes have one point in common, they have at least a second point in common.

Theorems—(i) Two intersecting planes cut one another in a straight line and in no points outside it. (ii) If a straight line is perpendicular to each of two intersecting straight lines at their point of intersection, it is perpendicular to the plane in which they lie. (iii) All straight lines drawn perpendicular to a given straight line at a given point on it are coplanar. (iv) If of two parallel straight lines one is perpendicular to a plane, the other is also perpendicular to it. (v) If a straight line is perpendicular to a plane, then every plane passing through it is also perpendicular to that plane.

Idea of the following regular solids: Sphere, rectangular parallelopiped, regular tetrahedron, right prism, right circular cylinder and a right cone. Expressions (without proof) for the surfaces and volumes of the above solids.

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SOLID GEOMETRY

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IMPORTANT FORMULÆ AND RESULTS

1. Distance between (x_1, y_1) and (x_2, y_2)

$$= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$

2. The point dividing the line joining (x_1, y_1) and (x_2, y_2) in the ratio of $m : n$ is

$$\left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}\right).$$

3. The area of the triangle with vertices (x_1, y_1), (x_2, y_2) and (x_3, y_3)

$$\triangle = \tfrac{1}{2}(x_1y_2 - x_2y_1 + x_2y_3 - x_3y_2 + x_3y_1 - x_1y_3)$$
$$= \tfrac{1}{2}\{x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\}.$$

Three points (x_1, y_1), (x_2, y_2) and (x_3, y_3) *are collinear*

$$\text{if } x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2) = 0$$

4. The equation of a straight line

(i) parallel to the y-axis is $x = l$.

(ii) parallel to the x-axis is $y = k$.

(iii) making intercepts a and b on the axes is

$$\frac{x}{a} + \frac{y}{b} = 1.$$

(iv) making an angle θ with the x-axis and cutting off intercept c from the y-axis is $y = mx + c$, where $m = \tan\theta$.

(v) in normal form is $x\cos\alpha + y\sin\alpha = p$.

(vi) of gradient m and through a given point (x_1, y_1) is

$$y - y_1 = m(x - x_1).$$

(vii) through two given pts. (x_1, y_1) and (x_2, y_2) is

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1).$$

(viii) in the symmetrical form is

$$\frac{y - y_1}{\sin\theta} = \frac{x - x_1}{\cos\theta} = r$$

5. The angle between the lines

(i) $y=m_1x+c_1$ and $y=m_2x+c_2$ is $\tan^{-1}\dfrac{m_1-m_2}{1+m_1m_2}$.

(ii) $a_1x+b_1y+c_1=0$ and $a_2x+b_2y+c_2=0$ is

$$\tan^{-1}\frac{a_1b_2-a_2b_1}{a_1a_2+b_1b_2}.$$

6. The lines

(i) $y=m_1x+c_1$ and $y=m_2x+c_2$ are parallel,

if $m_1=m_2$, and perpendicular, if $m_1m_2=-1$.

(ii) $a_1x+b_1y+c_1=0$ and $a_2x+b_2y+c_2=0$ are parallel,

if $\dfrac{a_1}{a_2}=\dfrac{b_1}{b_2}$, and perpendicular, if $a_1a_2+b_1b_2=0$.

7. Equation of a line through the intersection of the lines

$a_1x+b_1y+c_1=0$ and $a_2x+b_2y+c_2=0$ is

$a_1x+b_1y+c_1+k(a_2x+b_2y+c_2)=0$.

8. Length of a perpendicular from (x_1, y_1) on

$ax+by+c=0$ is $\dfrac{ax_1+by_1+c}{\sqrt{a^2+b^2}}$.

9. The equations of the bisectors of the angles between the lines $a_1x+b_1y+c_1=0$ and $a_2x+b_2y+c_2=0$ are

$$\frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}}=\pm\frac{a_2x+b_2y+c_2}{\sqrt{a_2^2+b_2^2}}.$$

10. For transformation of equations with respect to parallel xes through $(h, k,)$, put $x=x'+h$, $y=y'+k$.

11. The equation of a circle with centre (h, k) and radius is

$$(x-h)^2+(y-k)^2=r^2.$$

12. The general equation of a circle is $x^2+y^2+2gx+2fy+c=0$, whose centre is $(-g, -f)$ and radius is

$$\sqrt{g^2+f^2-c}.$$

13. The equation of a parabola with vertex as origin and the x-axis as axis of the parabola is

$$y^2=4ax,$$

where a is the distance of the focus from the vertex.

14. The equations of an ellipse and a hyperbola, with the axes of co-ordinates as the principal axes are respectively

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \text{ and } \frac{x^2}{a^2}-\frac{y^2}{b^2}=1.$$

15. (i) The equation of a tangent at (x_1, y_1) is obtained by putting xx_1 for x^2, yy_1 for y^2, $x+x_1$ for $2x$, $y+y_1$ for $2y$ in the equation of the curve.

(ii) The equation of the normal at (x_1, y_1)

to the circle $x^2+y^2=a^2$ is $xy_1-yx_1=0$;

to the parabola $y^2=4ax$ is $y-y_1=-\frac{y_1}{2a}(x-x_1)$;

to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, is $\frac{x-x_1}{x_1/a^2}=\frac{y-y_1}{y_1/b^2}$ and

to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is $\frac{x-x_1}{x_1/a^2}=\frac{y-y_1}{-y_1/b^2}$.

16. The equation to the normal to the parabola $y^2=4ax$ in terms of the gradient is $y=mx-2am-am^3$ with its foot at $(am^2, -2am)$.

The slope equation of the tangent to

(a) the circle $x^2+y^2=a^2$ is $y=mx\pm a\sqrt{1+m^2}$

(b) the parabola $y^2=4ax$ is $y=mx+\frac{a}{m}$.

(c) the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $y=mx\pm\sqrt{a^2m^2+b^2}$

(d) the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is $y=mx\pm\sqrt{a^2m^2-b^2}$

18. Length of the tangent from $(x' \; y')$ on a circle $x^2+y^2+2gx+2fy+c=0$ is $\sqrt{x'^2+y'^2+2gx'+2fy'+c}$.

Area of *triangle* $=\frac{1}{2}$ base $\times$ altitude.

$=\sqrt{s(s-a)(s-b)(s-c)}$

where $2s=a+b+c$ (a, b, c being the three sides)

Circumference of circle of radius $r=2\pi r$

Area ,, ,, $=\pi r^2$

Solid figures

Solid	Volume	Area of surface
Rectangular parallelopiped	abc	$2(ab+bc+ca)$
(Right) Prism	area of base $\times$ height	Lateral surface = perimeter of base $\times$ height Total surface = Lateral surface + area of the two end faces.
Right Pyramid	$\frac{1}{3}$ area of base $\times$ height	Lateral surface = sum of areas of the triangular faces $=\frac{1}{2}$ perimeter of base $\times$ slant height. Total surface = lateral surface + base area
Sphere	$\frac{4}{3}\pi r^3$	$4\pi r^2$
Right Circular Cylinder	$\pi r^2 h$	Curved surface $=2\pi rh$. Total surface $=2\pi rh+2\pi r^2$
Right Circular Cone	$\frac{1}{3}\pi r^2 h$	Curved surface $=\pi r\sqrt{h^2+r^2}$ Total surface $=\pi r\sqrt{h^2+r^2}+\pi r^2$

PRE-UNIVERSITY

CO-ORDINATE & SOLID GEOMETRY

CO-ORDINATE GEOMETRY

CHAPTER I

I-1. Co-ordinates.

The location of a point in a plane can be ascertained by its distances from two lines of reference.

Let $X'OX$ and YOY' be two fixed straight lines in the plane of paper. The line $X'OX$ is called the x-axis and

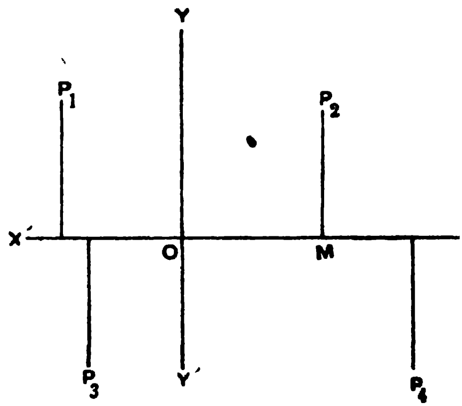


Fig. 1

line YOY' is called y-axis and two lines together are called the axes of reference.

The point O is called the origin of the co-ordinates or more shortly the origin. From the point P, draw PM parallel to the y-axis to meet the x-axis in M. The distance OM is called the abscissa and the distance MP, the ordinate of the point P, whilst the abscissa and the ordinate together are called its co-ordinates. If two lines OX, OY are at right angles, the axes OX, OY are called rectangular axes and its co-ordinates are called the rectangular Cartisian

system of co-ordinates. Lines measured in the direction OX are positive whilst those parallel to OX' are negative. Lines measured parallel to OY are positive whilst those parallel to OY' are negative. If the point P is in the quadrant XOY, x-co-ordinate (abscissa) is positive and y-co-ordinate (ordinate) is positive. If the point P_1 is in the quadrant $X'OY$, x-co-ordinate is negative and y-co-ordinate is positive.

If the point P_2 is in the quadrant $X'OY'$, both x and y co-ordinates are negative. If the point P_3 is in the quadrant XOY', x-co-ordinate is positive while the y-co-ordinate is negative. The co-ordinates of a point are written thus (x_1, y_1), $(1, 2)$, $(-1, 3)$ etc.

I-2. The distance between two points whose co-ordinates are given.

Let P_1 and P_2 be two given points and let (x_1, y_1) and (x_2, y_2) be co-ordinates of two points.

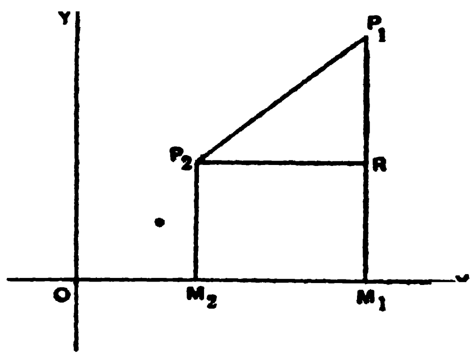


Fig. 2

Let P_1M_1, P_2M_2 be perpendiculars from P_1 and P_2 upon the line OX, then $P_1M_1 = y_1$, $OM_1 = x_1$, $OM_2 = x_2$ and $P_2M_2 = y_2$. Draw P_2R parallel to OX.

Then $P_2R = M_2M_1 = OM_1 - OM_2 = x_1 - x_2$

and $P_1R = P_1M_1 - P_2M_2 = y_1 - y_2$

In ΔP_1RP_2, $\angle P_1RP_2$ is a right angle

$\therefore \quad P_1P_2^2 = P_1R^2 + P_2R^2$

$$= (x_1 - x_2)^2 + (y_1 - y_2)^2$$

$\therefore$ *In rectangular co-ordinate the distance between two points* $= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

Cor. *The distance of a point* $(x_1 y_1)$ *from the origin*

$$= \sqrt{x_1^2 + y_1^2}$$

I-3. To find the co-ordinates of a point which divide a line joining 2 points (x_1, y_1) (x_2, y_2) in the ratio m : n.

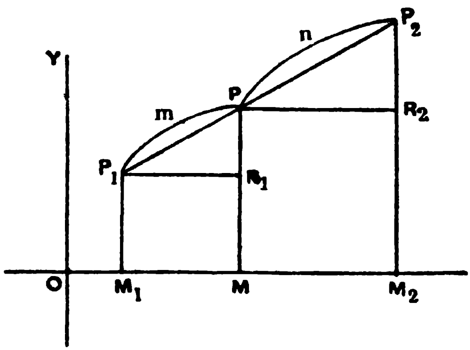


Fig. 8

Let P_1 be the point (x_1, y_1), P_2 the point (x_2, y_2) and let P be the required point (x, y) so that, we have

$$\frac{P_1P}{PP_2} = \frac{m}{n}$$

Draw P_1M_1, P_2M_2 be perpendiculars from two points P_1, P_2 upon x-axis

Then $P_1M_1 = y_1$, $OM_1 = x_1$

$P_2M_2 = y_2$, $OM_2 = x_2$

Draw P_1R_1 and PR_2 parallel to x-axis meeting PM, P_2M_2 in R_1 and R_2.

Also $P_1R_1 = M_1M = OM - OM_1 = (x - x_1)$

$$PR_1 = PM - P_1M_1 = y - y_1$$

$$PR_2 = MM_2 = OM_2 - OM = x_2 - x$$

$$P_2R_2 = P_2M_2 - PM = y_2 - y.$$

Since the triangles P_2PR_2 and PP_1R_1 are similar

hence, $\dfrac{PR_1}{P_2R_2} = \dfrac{P_1P}{PP_2} = \dfrac{m}{n}$

$$\therefore \quad \frac{y - y_1}{y_2 - y} = \frac{m}{n}$$

or, $\quad ny - ny_1 = my_2 - my$

or, $\quad my + ny = ny_1 + my_2$

$$\therefore \quad y = \frac{ny_1 + my_2}{m+n}$$

Similarly from the same two similar triangles

$$\frac{P_1R_1}{PR_2} = \frac{P_1P}{PP_2} = \frac{m}{n}$$

or $\quad \dfrac{x - x_1}{x_2 - x} = \dfrac{m}{n} \qquad$ or, $\quad nx - nx_1 = mx_2 - mx$

or, $\quad nx + mx = nx_1 + mx_2$

$$\therefore \quad x = \frac{nx_1 + mx_2}{m+n}$$

Cor. 1. *If the point Q divides the line P_1P_2, externally in the same ratio so that $P_1Q : QP_2 = m_1 : m_2$ the co-ordinate of Q would be found to be*

$$\frac{m_1x_2 - m_2x_1}{m_1 - m_2}, \quad \frac{m_1y_2 - m_2y_1}{m_1 - m_2}$$

Cor. 2. *The co-ordinates of the middle point of the line joining (x_1, y_1) to (x_2, y_2) are*

$$\frac{x_1 + x_2}{2}, \quad \frac{y_1 + y_2}{2}$$

I-4. The area of a trapezium whose parallel and AD are of length a and b respectively and h is the distance between two parallel sides. Draw AL perpendicular to BC and CN perpendicular to AD produced, then $AL = CN = h$.

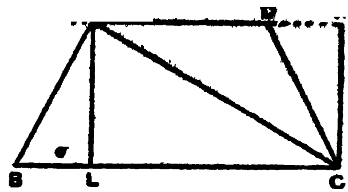


Fig. 4

Area of the trapezium

$$= \text{area of } \Delta ABC + \text{area of } \Delta ACD$$

$$= \tfrac{1}{2}\, BC \,.\, AL + \tfrac{1}{2}\, AD \,.\, CN$$

$$= \tfrac{1}{2}\, ah + \tfrac{1}{2}\, bh = \tfrac{1}{2}\,(a+b)h$$

I-5. *To find area of a triangle the co-ordinates of whose angular points are given.*

Let ABC be the triangle and let the co-ordinates of the vertices A, B, C be respectively (x_1, y_1) (x_2, y_2) (x_3, y_3).

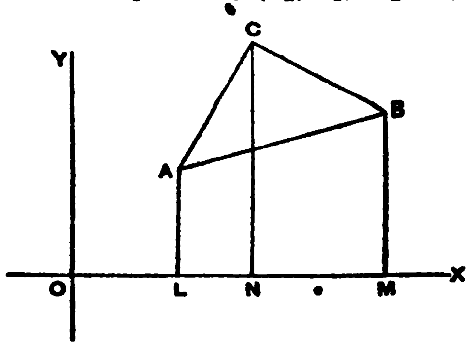


Fig. 5

Draw AL, BM, CN perpendiculars upon x-axis.

The area of triangle ABC = area of the trapezium $ALNC$ + area of the trapezium $CNMB$ − area of the trapezium $ALMB = \tfrac{1}{2}\, LN\ (AL+CN) + \tfrac{1}{2}\, NM\,(BM+CN)$

$$-\tfrac{1}{2}\, LM\,(AL+BM)$$

$$= \tfrac{1}{2}\,\{(x_3-x_1)\,(y_1+y_2)+(x_2-x_3)\,(y_2+y_3)$$

$$-\tfrac{1}{2}\,(x_2-x_1)\,(y_1+y_2)\}$$

$$= \tfrac{1}{2}\,\{(x_1y_2-x_2y_1)+(x_2y_3-x_3y_2)+(x_1y_2-x_2y_1)\}$$

Cor. 1. The area of triangle can be put in the determinant notation

$$=\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

Cor. 2. The area of the triangle whose vertices are $(0, 0)\ (x_1y_1)\ (x_2y_2)$ is $\frac{1}{2}(x_1y_2-x_2y_1)$

N. B. If we take the arrangement of vertices in counter clockwise order, the formula for expression of the area of the triangle in cyclic clockwise order of suffices 1, 2, 3 as in article I-5. But if we take the arrangement of the vertices in cyclic anti-clockwise order of the suffices and the area

$$=\frac{1}{2}\{(x_2y_1-x_1y_2)+(x_3y_2-x_2y_3)+(x_3y_1-x_1y_2)\}$$

I-6. *To find the area of a quadrilateral the co-ordinates of whose angular points are given.*

Let the angular points of the quadrilateral be A, B, C, D whose co-ordinates are $(x_1, y_1)\ (x_2, y_2)\ (x_3, y_3)$ and (x_4, y_4).

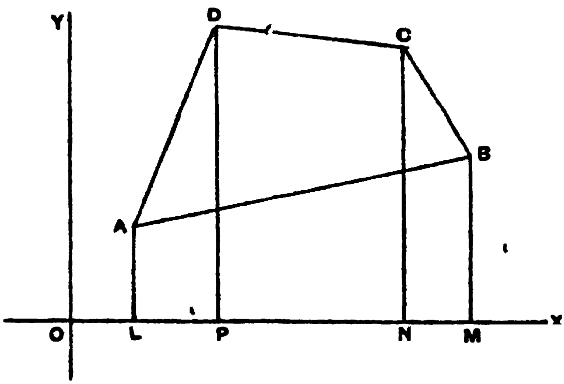


Fig. 6

Draw AL, BM, CN, DP perpendiculars on OX.

The area of quadrilateral

$$\begin{aligned}&=\text{trapezium } ALPD+\text{trapezium } DLNC\\&\quad+\text{trapezium } CNMB-\text{trapezium } ABML\\&=\tfrac{1}{2}\{(x_4-x_1)(y_1+y_4)+(x_3-x_4)(y_3+y_4)\\&\quad+(x_2-x_3)(y_2+y_3)-(x_2-x_1)(y_2+y_1)\}\\&=\tfrac{1}{2}\{(x_1y_2-x_2y_1)+(x_2y_3-x_3y_2)+(x_3y_4-x_4y_3)\\&\qquad+(x_4y_1-x_1y_4)\}\end{aligned}$$

Illustrative Examples

Ex. 1 Find the co-ordinates of the centroid of a triangle whose vertices are (x_1, y_1), (x_2, y_2), (x_3, y_3).

Hence prove that 3 medians of the triangle are concurrent.

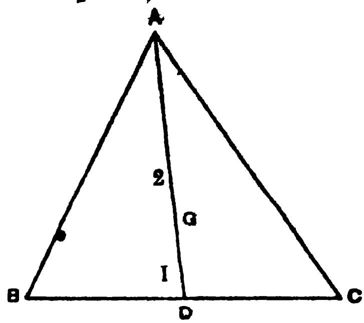


Fig. 7

Let $A\,(x_1 y_1)$, $B\,(x_2 y_2)$, $C\,(x_3 y_3)$ be vertices of the triangle. Let D be middle point of side BC.

Then the co-ordinates of D are

$$\left(\frac{x_2+x_3}{2}, \frac{y_2+y_3}{2}\right)$$

The centroid G divides AD in the ratio 2 : 1.

Hence the co-ordinates of G are

$$\frac{2\times\frac{x_2+x_3}{2}+x_1}{2+1}, \quad \frac{2\times\frac{y_2+y_3}{2}+y_1}{2+1}$$

$$\frac{x_2+x_3+x_1}{3}, \quad \frac{y_2+y_3+y_1}{3}$$

The symmetry of this co-ordinates shows that if we take other medians BE, CF and divide them in the ratio 2 : 1, then we get the same co-ordinates. So that the point of trisection of each median is the same and hence 3 medians meet at a point called centroid

Ex. 2. If $(x_1 y_1)$, $(x_2 y_2)$, $(x_3 y_3)$, $(x_4 y_4)$ be angular points of a parallelogram, then $x_1+x_3=x_2+x_4$ and $y_1+y_3=y_2+y_4$.

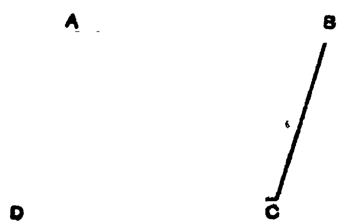


Fig. 8

Denote the angular points by A, B, C, D. Since diagonals of a parallelogram bisect each other, the middle point of the diagonal AC is the same point as the middle point of the diagonal BD.

Now co-ordinates of the middle point of AC are

$$\left(\frac{x_1+x_3}{2}, \frac{y_1+y_3}{2}\right)$$

Also the co-ordinates of the middle point of BD are

$$\left(\frac{x_2+x_4}{2}, \frac{y_2+y_4}{2}\right)$$

$$\therefore \quad \frac{x_1+x_3}{2}=\frac{x_2+x_4}{2} \text{ and } \frac{y_1+y_3}{2}=\frac{y_2+y_4}{2}$$

Hence $x_1+x_3=x_2+x_4$; $y_1+y_3=y_2+y_4$.

Ex. 3. Show that the three points (4, 2), (7, 5), (9, 7) lie on a straight line.

The area of the triangle formed by these 3 points

$$=\tfrac{1}{2}\,[(4.5-2.7)+(7.7-9.5)+(9.2-7.4)]$$

$$=\tfrac{1}{2}\,[6+4-10]=0.$$

Since the area of the triangle formed by the three points is zero, the three points are on a right line.

Ex. 4. In any triangle ABC, prove that

$AB^2+AC^2=2(AD^2+BD^2)$ where D is the middle point of BC.

Let $(0, 0)$, $(x_1 y_1)$, $(x_2 y_2)$ be the points B, A, C.

Then D, the middle point of BC has co-ordinates

$$=\left(\frac{0+x_2}{2}, \frac{0+y_2}{2}\right)=(\tfrac{1}{2}x_2, \tfrac{1}{2}y_2)$$

$$AB^2=(0-x_1)^2+(0-y_1)^2=x_1^2+y_1^2$$

$$AC^2=(x_1-x_2)^2+(y_1-y_2)^2=x_1^2+y_1^2+x_2^2+y_2^2 -2x_1x_2-2y_1y_2$$

$$AB^2+AC^2=2\,[x_1^2+y_1^2]+x_2^2+y_2^2-2x_1x_2-2y_1y_2$$

$$AD^2=\left(x_1-\frac{x_2}{2}\right)^2+\left(y_1-\frac{y_2}{2}\right)^2$$

$$=x_1^2+y_1^2+\tfrac{1}{4}(x_2^2+y_2^2)-x_1x_2-y_1y_2$$

$$BD^2=\left(0-\frac{x_2}{2}\right)^2+\left(0-\frac{y_2}{2}\right)^2=\tfrac{1}{4}(x_2^2+y_2^2)$$

$$AD^2+BD^2=x_1^2+y_1^2+\tfrac{1}{2}(x_2^2+y_2^2)-x_1x_2-y_1y_2$$

$$2(AD^2+BD^2)=2[x_1^2+y_1^2]+x_2^2+y_2^2-2x_1x_2-2y_1y_2$$

$$\therefore\quad AB^2+AC^2=2(AD^2+BD^2)$$

Ex. 5. Show that the area of the quadrilateral whose angular points taken in order are $(a, 0)$, $(-b, 0)$, $(0, a)$, $(0, -b)$ is zero. Give the justification of this result.

Denote the points by $ABCD$. Area of the quadrilateral

$$=\tfrac{1}{2}\,[\{a\,.\,0-0(-b)\}+(-b.\,a-0.\,0)$$

$$+(0.\,-b\,-a\,.\,0)+\{0\,.\,0-(-b)a\}]=\tfrac{1}{2}[-ab+ab]=0.$$

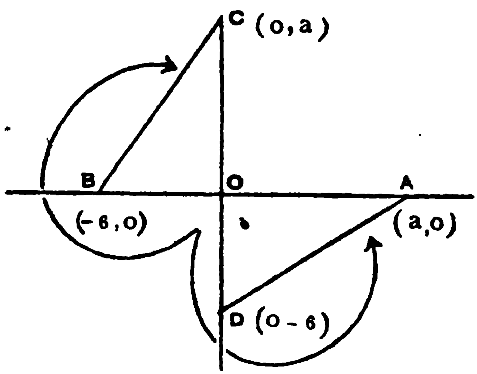


Fig. 9

Now, the figure enclosed by these 4 points encloses 2 triangles OBC and ODA. Since the arrangement of vertices of $\triangle OBC$ is in clockwise-direction while the arrangement of vertices of $\triangle ODA$ is in anti-clockwise direction. Also magnitude of $\triangle OBC$ and that of $\triangle ODA$ are equal in magnitude $(=\tfrac{1}{2}ab)$. $\therefore$ The algebraic sum of areas of 2 triangles is zero.

Exercises 1

1. Find the distances between these following points :
   (i) $(-3, -2)$ and $(-6, 7)$ (ii) $(a, 0)$ and $(0, b)$
   (iii) $(at_1^2,\ 2at_1)$ and $(at_2^2,\ 2at_2)$.

2. Find the value of x_1, if the distance between the points $(x_1, 2)$ and $(3, 4)$ be 3.

3. Show that the points $(2a, 4a)$, $(3a, 6a)$, $(2a+\sqrt{3}a, 5a)$ are the angular points of a equilateral triangle.

4. Prove that the points $(2, -2)(8, 4)$ $(5, 7)$ and $(-1, 1)$ are angular points of a rectangle.

5. Prove that the straight line joining the middle points of two adjacent sides is parallel to the third side.

6. Prove that the point $(1, 2)$ is the centroid of the triangle formed by $(0, 0)$, $(1, 2)$, $(2, 4)$

7. Prove that the condition that three points $(\alpha_1\beta_1)$, $(\alpha_2\beta_2)$ and $(\alpha_3\beta_3)$ are collinear of

$$(\alpha_1\beta_2-\alpha_2\beta_1)+(\alpha_2\beta_3-\alpha_3\beta_2)+(\alpha_3\beta_1-\alpha_1\beta_3)=0.$$

8. Find co-ordinates of a point which divides the joining of points $(1, 3)$ and $(2, 7)$ in the ratio $3:4$.

9. Prove that the following sets of 3 points are collinear :

(i) $(-4, 4)$, $(2, 0)$ and $(5, -2)$

(ii) $(1, 4)$, $(3, -2)$ and $(-3, 16)$

(iii) $(a, b+c)$, $(b, c+a)$ and $(c, a+b)$

10. Find the area of triangle formed by following sets of points :

(i) $(0, 4)$, $(3, 6)$ and $(-8, 2)$

(ii) $(a, c+a)$ (a, c) $(-a, c-a)$

(iii) $(at_1^2, 2at_1)(2at_2^2, 2at_2)(at_3^2, 2at_3)$

11. Find the area of the quadrilateral formed by the following sets of points :

(i) $(1, 1)(3, 4)$, $(5, -2)$ and $(4, -7)$

(ii) $(-1, 6)(-3, -9)(5, -8)$ and $(3, 9)$

12. The co-ordinates of A, B, C are $(6, 3)(-3, 5)$ and $(4, -2)$ respectively and p is the point (x, y)

Show that $\dfrac{\Delta PBC}{\Delta ABC}=\dfrac{x+y-2}{7}$ [C.U.]

13. Prove that the figure formed by joining $(2, 4)(4, 6)$, $(8, 10)$ and $(6, 8)$ in order is a parallelogram.

14. Find the co-ordinates of centre of the circle circumscribing the triangle whose vertices are $(1, 1)(2, 3)$ and $(-2, 2)$

15. Prove that the lines joining the middle points of opposite sides of a quadrilateral and the line joining the middle points of its diagonals meet at a point and bisect one another. [C. U.]

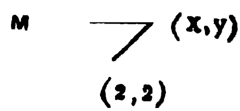
CHAPTER II

II-1. When a point moves so as to satisfy a given condition or conditions, the path, it traces out is called the locus under the conditions. If the co-ordinates of the variable point $P(x, y)$, satisfying the given condition or conditions, satisfy an equation, the equation is said to be the equation of the locus. For example, if the variable point P moves in such a manner that its distance from a fixed point (a, b) is always equal to r then $(x-a)^2+(y-b)^2=r^2$. The relation between the co-ordinates of the variable point is the equation of the locus.

Illustrative Examples

Ex. 1. Find the locus of a point which moves so that its distance from the axis of y is double its distance from the point (2, 2). [C. U.]

Let $P(x, y)$ be the variable point, so that its distance from y-axis $=x$.



The distance of point P from the point (2, 2)

$$=\sqrt{(x-2)^2+(y-2)^2}$$

$$\therefore \quad x=2\sqrt{(x-2)^2+(y-2)^2}$$

Fig. 10

$$\therefore \quad x^2=4\left[(x-2)^2+(y-2)^2\right]$$

$$x^2=4x^2-16x+16+4y^2-16y+8$$

Ex. 2. Find the equation of the perpendicular bisector of the join of the points (2, 4) (6, 12).

Let $P(x, y)$ be a variable point of the perpendicular bisectors of $A(2, 4)$; $B(6, -2)$

Then P is equidistant from A and B

Now $PA^2=(x-2)^2+(y-4)^2$

$$PB^2=(x-6)^2+(y+2)^2$$

$\therefore$ The required equation is

$$(x-2)^2+(y-4)^2=(x-6)^2+(y+2)^2$$

or, $-4x+4-8y+16=-12x+4y+36+4$

or, $8x-12y-20=0$

or, $4x-3y-5=0$

Exercises 2

1. Find the equation of the locus of a point which moves so that.

(*i*) Its distance from y-axis is less than its distance from x-axis by 2.

(*ii*) Its distance from x-axis = its distance from y-axis.

(*iii*) Its distance from the origin is equal to its distance from the point (a, b).

(*iv*) The sum of the squares of its distances from $(1, 0)$ and $(0, 1)$ is always equal to 10.

(*v*) The sum of its distances from two straight lines at right angle is equal to a.

CHAPTER III

THE STRAIGHT LINE

III-1. Any equation of first degree in x and y represents a straight line which can satisfy any two given conditions. The most general form of the first degree equation is $ax+by+c=0$ of which a, b, c are constants and two of these are effective constants because we can divide the equation by c and the equation can take up the form $ln+my=1$. Hence these are two arbitary constants in the equation and the form of the equation is definite when the straight line satisfies 2 specified conditions.

III-2. The equation of straight lines parallel to x-axis can be put in the form $y=k$ for y-co-ordinates of all points of such straight line are equal to k. Similarly, the equation of straight line parallel to y-axis can be put in the form $x=l$ for the distances of all points on the line from y-axis are equal to l.

In particular $y=0$ is the equation of x-axis

and $x=0$ is the equation of y-axis.

III-3. **Intercept form** of the equation of the straight line.

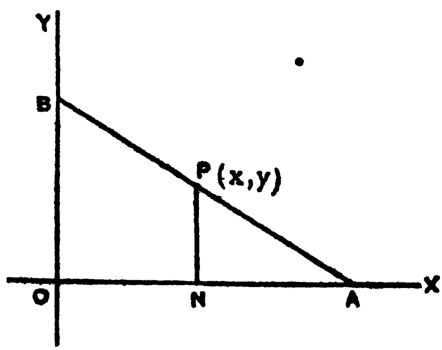


Fig. 11

Let AB be the straight line whose intercepts on x-axis and y-axis are given to be of lengths a and b.

Let $P\ (x, y)$ be a variable point on the line and let PN be drawn perpendicular from P on x-axis.

So that $ON = x$, $PN = y$

Now Δ^s AOB, ANP are similar,

$$\frac{PN}{OB} = \frac{AN}{OA}$$

$$\text{or,}\quad \frac{y}{b} = \frac{a-x}{a}$$

$$\text{or,}\quad \frac{y}{b} = \frac{a}{a} - \frac{x}{a}$$

$$\therefore \frac{x}{a} + \frac{y}{b} = 1$$

N. B. Intercepts on x-axis are considered to be negative when the straight line cuts x-axis on the left side of O *i.e.*, the line OX'. The intercepts on y-axis are considered to be negative when the straight line cuts the y-axis on the downwards of O *i.e.*, the line OY'.

We can reduce the equation of any straight line in the intercept form *e. g.* $ax+by+c=0$ can be put in the form

$$\frac{x}{\frac{-c}{a}} + \frac{y}{\frac{-c}{b}} = 1$$

showing the intercepts on x-axis and y-axis are

$$-\frac{c}{a},\ -\frac{c}{b}$$

The equation $3x-4y-12=0$ can be put in the form $\frac{x}{4} + \frac{y}{-3} = 1$ showing that the intercepts on axes are 4 and -3.

III-4. Gradient of a straight line and the gradient form.

The gradient of a straight line means the increase of the ordinate per unit length of abscissa of the moving point.

Let P_1, P_2, P_3 be moving points on a straight line AB

Let P_1N, P_2M, P_3L be perpendiculars from P_1, P_2, P_3 upon x-axis.

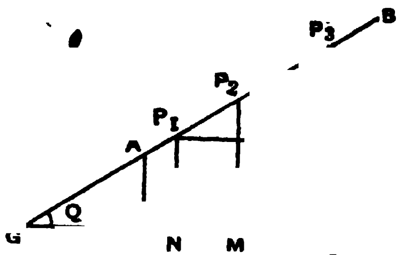


Fig. 12

P_2K = increase of ordinate as the point moves from P_1 to P_2

$$\frac{P_2K}{P_1K} = \text{the gradient of the line} = \frac{P_2M}{GM}$$

$= \tan\theta$, when θ is angle,

which the straight line makes with x-axis m generally denotes the value of gradient of the line $AB = \tan\theta$

III-5. *To find the equation of the straight line which makes an angle θ with x-axis and makes intercept c on y-axis.*

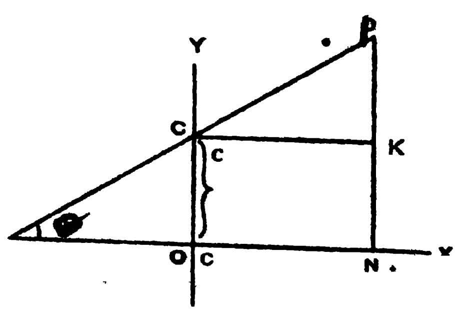


Fig. 13

Let P (x, y) be a moving point on the straight line. Draw PN perpendicular to x-axis.

Now $ON=CK=x$

$PK=PN-OC$

$=y-c$

Now $\frac{PK}{CK}=\tan PCK$

$=\tan\theta$

$\therefore \quad \frac{y-c}{x}=\tan\theta$

or, $y-c=x\tan\theta$

giving $y=x\tan\theta+c$

We denote $\tan\theta=m=$gradient of the straight line. Hence the equation of straight line in the gradient form is

$y=mx+c.$

Cor. 1. If the straight line passes through origin, $c=$the intercept on y-axis$=0$

The equation becomes $y=mx$

III-6. To find the equation to the straight line of given gradient m and passing through the given point (x_1, y_1).

Let the equation of the straight line be $y=mx+c.$······(1)

Since the straight line passes through $x_1 y_1$

$y_1=mx_1+c$······(II)

Subtracting (II) from (I)

$y-y_1=m(x-x_1)$

III-7. To find the equation to the straight line which passes through two given points $(x_1 y_1)$ and $(x_2 y_2)$.

Let the equation to the straight line be

$y=mx+c$·········(1)

As the straight line passes through $x_1 y_1$

$y_1=mx_1+c$·········(2)

Also as the straight line passes through $x_2 y_2$

$y_2=mx_2+c$.........(3)

Subtracting (2) from (1) $y-y_1=m(x-x_1)\cdots\cdots(4)$

" (2) " (3) $y_2-y_1=m(x_2-x_1)$

$$\therefore \quad m=\frac{y_2-y_1}{x_2-x_1}$$

Putting the value of m in the equation (4) we get the equation to the line

$$y-y_1=\frac{y_2-y_1}{x_2-x_1}(x-x_1)$$

III-8. To find the equation to the straight line in terms of the perpendicular drawn from origin upon the straight line and the angle which this perpendicular makes with the axis of x

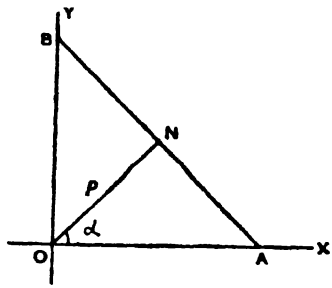


Fig. 14

Let the equation of the straight line be

$$\frac{x}{a}+\frac{y}{b}=1\cdots\cdots\cdots\cdots(I)$$

where a = intercept on x-axis $=OA$

and b = intercept on y-axis $=OB$

Let ON = length of perpendicular from O upon the straight line $=p$ and $\angle AON=\alpha$

Now $\frac{p}{OA}=\cos\alpha \quad \therefore \quad OA=\frac{p}{\cos\alpha} \quad \therefore \quad a=\frac{}{\cos\alpha}$

$$\frac{p}{OB}=\cos BON=\cos(90-\alpha)=\sin\alpha$$

$$\therefore \quad OB=\frac{p}{\sin\alpha}$$

$$\text{or,} \quad b=\frac{p}{\sin\alpha}$$

$$\therefore \text{ From (I) } \frac{x}{\frac{p}{\cos\alpha}}+\frac{y}{\frac{p}{\sin\alpha}}=1$$

$$\frac{x\cos\alpha}{p}+\frac{y\sin\alpha}{p}=1$$

$\therefore$ The equation to the straight line is

$x \cos \alpha + y \sin \alpha = p$

III-9. The symmetrical form of the equation to the straight line passing through a given point $(x_1 y_1)$

Let θ be slope of the line passing through $(x_1 y_1)$

Let the equation to the line be $y = mx + c \cdots\cdots\cdots(1)$

As it passes through $(x_1 y_1)$, $y_1 = mx_1 + c \cdots\cdots\cdots(2)$

Subtracting (2) from (1) $y - y_1 = m\,(x - x_1) \cdots\cdots(3)$

$$\text{Now } m = \tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$\text{from (3) } y - y_1 = \frac{\sin\theta}{\cos\theta}(x - x_1)$$

$$\frac{y - y_1}{\sin\theta} = \frac{x - x_1}{\cos\theta} = r$$

where r is the distance of any point (x, y) from the fixed point $(x_1\ y_1)$

III-10. Trace the following straight lines

(I) $2x - 3y = 6$

Converting this equation thus $\frac{2x}{6} - \frac{3y}{6} = 1$

$$\frac{x}{3} + \frac{y}{-2} = 1$$

This form shows that intercept on x-axis is 3 and the intercept on y-axis $= -2$

or, we can reduce this equation thus

$$-3y = -2x + 6$$

$$\text{or, } y = \frac{2}{3}x - 2$$

Fig. 15

Showing that the intercept on y-axis is -2 and the gradient of the straight line is $\frac{2}{3}$.

(II) $8x+7y+28=0$

Reducing the equation in the form

$$\frac{8x}{-28}+\frac{7y}{-28}-1=0$$

or, $\dfrac{x}{-\frac{7}{2}}+\dfrac{y}{-4}=1$ showing that intercepts on axes are $-\frac{7}{2}, -4$.

Also we can reduce the equation in m-form

$$7y=-8x-28$$

$$y=-\tfrac{8}{7}x-4$$

Showing that the gradient of the line $-\frac{8}{7}$ and intercept on y-axis -4.

ILLUSTRATIVE EXAMPLES

Ex. 1. Find the equation to the straight line which passes through (5, 3) and cut off equal positive intercepts from the axes.

Let the equation to the straight line having equal positive intercepts a on axes be $\frac{x}{a}+\frac{y}{a}=1$

As the straight line passes through (5, 3)

$$\frac{5}{a}+\frac{3}{a}=1 \qquad \text{or,} \quad \frac{8}{a}=1 \text{ giving } a=3$$

$\therefore$ The equation to the straight line is $x+y=8$.

Ex. 2. Find the equation to the straight line which passes through the given point (x', y') and is such that the given point bisects the part intercepted between the axes.

Let the equation to the straight line be $\frac{x}{a}+\frac{y}{b}=1$ and it cuts axes in points (a, o) and (o, b).

The middle point of join of two points (a, o) and (o, b) is

$\left(\frac{a}{2}, \frac{b}{2}\right)$ so that $x'=\frac{a}{2}$, $y'=\frac{b}{2}$ giving $a=2x'$ and $b=2y'$

Hence the equation to the line becomes

$$\frac{x}{2x'}+\frac{y}{2y'}=1$$

Ex. 3. Verify that three points (1, 5), (3, 14) and (−1, −4) are collinear. Also find the line of collinearity.

[C. U. 1957]

Equation to the line joining (1, 5), (3, 14) is

$$y-5=\frac{14-5}{3-1}\left(x-1\right)$$

$$\text{or,}\quad 2(y-5)=9(x-1)$$

$$\text{or,}\quad 9x-2y+1=0$$

If we put $x=-1$, $y=-4$

$$-9-2(-4)+1=0$$

∴ The equation to the line is satisfied by co-ordinates of the point (−1, −4)

∴ Three points (1, 5) (3, 14) (−1, −14) are collinear and the equation to the line is $9x-2y+1=0$

Ex. 4 Show that the distance of the point (x_0, y_0) from the line $ax+by+c=0$ measured parallel to a line making angle θ with x-axis is

$$-\frac{ax_0+by^0+c}{a\cos\theta+b\sin\theta}$$

Let the equation to the straight line through $(x_0\ y_0)$ making angle θ with x-axis be

$$\frac{x-x_0}{\cos\theta}=\frac{y-y_0}{\sin\theta}=r$$ where r is distance of any point (x, y) from $x_0\ y_0$

$$\therefore\quad x=x_0+r\cos\theta,\ y=y_0+r\sin\theta$$

If the point (x, y) be on the given straight line

$ax+by+c=0$, we have

$$a(x_0+r\cos\theta)+b(y_0+r\sin\theta)+c=0$$

$$r(a\cos\theta+b\sin\theta)=-(ax_0+by_0+c)$$

$$r=-\frac{ax_0+by_0+c}{a\cos\theta+b\sin\theta}.$$

Exercises 3 (A)

1. Find the equations to the straight lines passing through the following pairs of points

(a) (3, 4) and (5, 6) (b) (0, $-a$) and (b, 0)

(c) (at_1^2, $2at_1$) (at_2^2, $+2at_2$) (d) (-1, 3) and (4, -2)

2. Find the equation to the straight line cutting off an intercept -5 from the axis of y and being equally inclined to the axes.

3. Find the equation to the straight line cutting off an intercept 2 from negative direction of y and inclined at 30° with OX.

4. Find the equation to the straight line which passes through the point (-5, 4) and is such that the portion of it between the axes is divided by the point in the ratio 1 : 2.

5. Find the equation to the straight line which passes through the point (-4, 3) and is such that the portion of it between the axes is divided by the point in the ratio 5 : 3.

6. Find the equation to the sides of the triangles, the co-ordinates of whose angular points are respectively

(i) (1, 4) (2, -3) and (-1, -2)

(ii) (0, 1) (2, 0) and (-1, -2)

7. Find the equation to the diagonals of the rectangle the equations of whose sides are $x=a$, $x=a'$, $y=b$ and $y=b'$.

8. If the points (a, b), (a', b'), $(a-a', b-b')$ are collinear show that their join passes through the origin and that $ab^1=a^1b$.

[C. U].

9. Find the equation to the straight line passing through the point (1, 2) and the middle point of the join of (3, -4) and (5, -6).

10. A straight line passes through the point (-2, 3) and has a gradient $\frac{1}{2}$, find its equation. Find also the intercepts cut off by the line from the axes.

11. Reduce the following equations to the perpendicular form and hence find the length of perpendicular from the origin

(i) $x+y+2=0$ (ii) $x+\sqrt{3}y+14=0$.

12. Find the equation to the straight line which is inclined to an angle 45° to the axis of x and bisects the join of the points (4, 7) and (6, 5).

III-11. **Angle between two straight lines.**

(i) Let the equations to the lines be

$$y=m_1x+c_1\cdots(1) \text{ and}$$
$$y=m_2x+c_2\cdots(2)$$

and let θ be angle between the two straight lines PA, PB.

Let θ_1 and θ_2 be the angles which the straight lines AP, BP make with x-axis.

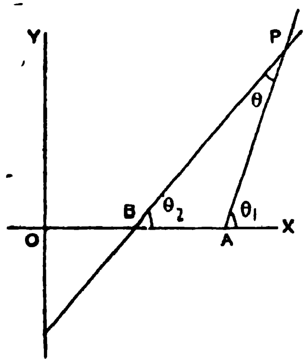


Fig. 16

Then $\tan\theta_1=m_1$

and $\tan\theta_2=m$

Now from ΔAPB, $\theta_1=\theta+\theta_2$

$$\therefore\quad \theta=\theta_1-\theta_2$$

$$\therefore\quad \tan\theta=\tan(\theta_1-\theta_2)=\frac{\tan\theta_1-\tan\theta_2}{1+\tan\theta_1\tan\theta_2}$$

$$=\frac{m_1-m_2}{1+m_1m_2}$$

(ii) If the equations to the lines be

$$a_1x+b_1y+c_1=0$$

$a_2x+b_2y+c_2=0$, we can write these equations in gradient or m-form thus,

$$-\frac{a_1}{b_1}x-\frac{c_1}{a_1}$$

$$\text{and } y\quad -\frac{a_2}{b_2}x-c_2$$

Thus we have $m_1\ -\frac{a_1}{b_1}$, $m_2=-\frac{a_2}{b_2}$

where m_1 and m_2 are the respective gradient of the two lines.

Hence $\tan\theta = \dfrac{m_1 - m_2}{1 + m_1 m_2} = \dfrac{\left(-\dfrac{b_1}{a_1}\right) - \left(-\dfrac{b_2}{a_2}\right)}{1 + \left(-\dfrac{b_1}{a_1}\right)\left(-\dfrac{b_2}{a_2}\right)}$

$$\therefore \quad \tan\theta = \frac{b_2 a_1 - a_2 b_1}{a_1 a_2 + b_1 b_2}$$

(iii) If the equations of the lines are given in the form

$$x \cos \alpha_1 + y \sin \alpha_1 = p_1$$
$$x \cos \alpha_2 + y \sin \alpha_2 = p_2$$

where α_1, α_2 are the angles which the perpendiculars from the origin make with x-axis.

Here clearly $\theta = \alpha_1 - \alpha_2$ or $180 - (\alpha_1 - \alpha_2)$

III-12. The condition of parallelism and perpendicularity of two lines.

(i) If the lines are parallel, $\theta_1 = \theta_2$

$$\therefore \quad \tan\theta_1 = \tan\theta_2$$
$$m_1 = m_2$$

and $-\dfrac{a_1}{b_1} = -\dfrac{a_2}{b_2}$

giving $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \dfrac{1}{k}$, say so that $a_2 = a_1 k$ and $b_2 = b_1 k$.

Thus the equation of two parallel lines $a_1 x + b_1 y + c_1 = 0$

and $a_2 x + b_2 y + c_2 = 0$ can be put in the forms

$a_1 x + b_1 y + c_1 = 0$ and $a_2 x + b_2 y + c_2 = 0$

i. e., $a_1 k x + b_1 k y + c_2 = 0$

or, $a_1 x + b_1 y + c_1 = 0$ and $a_1 x + b_1 y + \dfrac{c_2}{k} = 0$

$\therefore$ Two equations to the parallel straight lines differ by constant terms.

(ii) If the straight lines are perpendicular, then $\theta = 90$

$$\therefore \quad \frac{m_1 - m_2}{1 + m_1 m_2} = \tan 90^\circ = \infty$$

$$\therefore \quad 1 + m_1 m_2 = 0 \text{ giving } m_1 m_2 = -1$$

$$\text{Also } m_1 m_2 = \left(-\frac{a_1}{b_1}\right)\left(-\frac{a_2}{b_2}\right) = -1$$

$$\text{giving } a_1 a_2 + b_1 b_2 = 0.$$

Hence, to obtain the equation of a straight line perpendicular to a given straight line, either interchange the coefficients of x and y or make their coefficients reciprocal and then change the sign of either. The resulting equation represents a line perpendicular to the given one.

$$\text{Thus } a_1 x + b_1 y + c_1 = 0$$

$$b_1 x - a_1 y + c_2 = 0 \text{ or, } \frac{x}{a_1} - \frac{y}{b_1} + c_2 = 0$$

are the equations to two perpendicular lines for

$$a_1 b_1 - b_1 a_1 = 0 \text{ or, } a_1 \cdot \frac{1}{a_1} + b_1\left(-\frac{1}{b_1}\right) = 0$$

Ex. Find equations to lines (i) parallel to $3x+5y+7=0$ and (ii) perpendicular to $3x+5y+7=0$.

The equations of straight lines parallel to $3x+5y+7=0$ can be written by changing the constant term $3x+5y+k=0$.

The equations of straight lines perpendicular to $3x+5y+7=0$ can be written by the former rule thus

$$5x - 3y + k = 0.$$

III-13. Point of intersection of two straight lines

Let the equations to two straight lines

$$a_1 x + b_1 y + c_1 = 0$$
$$a_2 x + b_2 y + c_2 = 0$$

Since the point of intersection of two straight lines is the point common to 2 straight lines and let it be (α, β) so that the co-ordinates (α, β) must satisfy these equations

$$a_1\alpha+b_1\beta+c_1=0 \quad \cdots \quad \cdots \quad (1)$$

$$a_2\alpha+b_2\beta+c_2=0 \quad \cdots \quad \cdots \quad (2)$$

$\therefore$ By rule of cross multiplication

$$\frac{\alpha}{b_1c_2-b_2c_1}=\frac{\beta}{c_1a_2-c_2a_1}=\frac{1}{a_1b_2-a_2b_1}$$

giving $\alpha=\dfrac{b_1c_2-b_2c_1}{a_1b_2-a_2b_1}$, $\beta=\dfrac{c_1a_2-c_2a_1}{a_1b_2-a_2b}$.

N. B. If $a_1b_2-a_2b_1=0$, co-ordinates of points of intersection become infinite and in this case, two straight lines intersect at a point lying at infinite distance.

In this case, two straight lines are parallel because parallel straight lines meet at infinity.

$$a_1b_2-a_2b_1=0 \text{ gives}$$

$\dfrac{a_1}{a_2}=\dfrac{b_1}{b_2}$ which is the condition that two lines are parallel.

III-14. Condition of concurrence of three given straight lines

$$a_1x+b_1y+c_1=0\cdots\cdots(1)$$

$$a_2x+b_2y+c_2=0\cdots\cdots(2)$$

$$a_3x+b_3y+c_3=0\cdots\cdots(3)$$

The co-ordinates of the point of intersection of lines (1) and (2) is given by III-11

$$\frac{b_1c_2-b_2c_1}{a_1b_2-a_2b_1}, \frac{c_1a_2-c_2a_1}{a_1b_2-a_2b_1}$$

If three lines meet at a point, this point must lie on line (3) and this co-ordinates of the point of intersection of lines (1) and (2) must satisfy (3)

Hence $a_3\dfrac{b_1c_2-b_2c_1}{a_1b_2-a_2b_1}+b_3\dfrac{c_1a_2-c_2a_1}{a_1b_2-a_2b_1}+c_3=0$

$$a_3(b_1c_2-b_2c_1)+b_3(c_1a_2-c_2a_1)+c_3(a_1b_2-a_2b_1)=0$$

This condition can be expressed in the determinant form

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

III-15. Equation of straight lines passing through the intersection of two given lines

Let $a_1x+b_1y+c_1=0 \cdots\cdots(1)$

$a_2x+b_2y+c_2=0 \cdots\cdots(2)$ be equations of two lines.

If we consider the following equation

$$a_1x+b_1y+c_1+k(a_2x+b_2y+c_2)=0 \cdots\cdots(3)$$

we see that this is an equation of the first degree representing a straight line. Also if (α, β) be co-ordinates of the point of intersection of the straight lines (1) and (2) then $a_1\alpha+b_1\beta+c_1=0$ and $a_2\alpha+b_2\beta+c_2=0$ and hence $a_1\alpha+b_1\beta+c_1+k(a_2\alpha+b_2\beta+c_2)\equiv 0$ for all values of k.

Therefore the equation (3) represents a straight line passing through the point of intersection of straight lines (1) and (2) for all values of k.

Conversely the equation $a_1x+b_1y+c_1+k(a_2x+b_2y+c_3)=0 \cdots(3)$ represents straight lines passing through the point of intersection of two lines whose equations are $a_1x+b_1y+c_1=0$ and $a_2x+b_2y+c_2=0$

Hence all straight lines represented by the equation (3) always pass through the fixed point of intersection of two straight lines (1) and (2).

ILLUSTRATIVE EXAMPLES

Ex. 1. Find the equation to the right line passing through the origin which is

(i) parallel to line $ax+by+c=0$

(ii) perpendicular to line $ax+by+c=0$

(i) The equation to the right line parallel to the line $ax+by+c=0$ can be written as

$ax+by+k=0$ (for the equations to parallel lines differ only by the constant terms.)

Now because this straight line passes through origin this equation is satisfied by $x=0$, $y=0$

$\therefore \quad a\,.\,0+b\,.\,0+k=0$ giving $k=0$

$\therefore$ The equation to the required line

$$ax+by=0$$

(ii) The equation of the straight line perpendicular to the line $ax+by+c=0$ can be written as $bx-ay+k=0$

[Art. III-10 (ii)]

Now because this straight line passes through the origin, this equation is satisfied by $x=0$, $y=0$

$\therefore \quad b\,.\,0-a\,.\,0+k=0$ giving $k=0$

$\therefore$ The equation of the required line

$$bx-ay=0$$

Ex. 2. Find the equation to the straight line which is perpendicular bisector of join of (3, 4) and (5, 6)

The equation of line joining two points (3, 4) and (5, 6)

$$y-4=\frac{6-4}{5-3}(x-3)$$

$$y-4=(x-3)\cdots\cdots(1)$$

Also the co-ordinates of the middle point of (3, 4) and (5, 6)

are $\left(\frac{3+5}{2}, \frac{4+6}{2}\right)$ or, $(4, 5)$

Now the equation of line through (4, 5) can be written as

$$y-5=m(x-4)\cdots\cdots(2)$$

The condition that lines (1) and (2) are perpendicular $1\,.\,m=-1$ giving $m=-1$

The equation of required line can be obtained by putting $m=-1$ in the equation (2)

$$y-5=-(x-4)$$
$$x+y=9$$

Ex. 3. Prove that three lines, $x-y-7=0$, $x+2y+6=0$ and $2x+y-1=0$ pass through a common point and this point is equidistant from $(5, -4)$, $(3, -2)$ and $(1, -6)$

The co-ordinates of the point of intersection of

$$x-y-7=0\cdots(1)$$

and $x+2y+6=0\cdots(2)$ can be obtained by solving.

Subtracting (2) from (1) $-3y-13=0$ $y=-\frac{13}{3}$, $x=\frac{8}{3}$

Now the equation $2x+y-1=0$ is satisfied by

$$x=\tfrac{8}{3},\ y=-\tfrac{13}{3} \quad \because \quad \tfrac{16}{3}-\tfrac{13}{3}-3=0$$

$\therefore$ The three straight lines meet at $(\frac{8}{3}, -\frac{13}{3})$

Ex. 4. Find the co-ordinates of the foot of perpendicular from the point (2, 3) on the line $x+y-11=0\cdots(1)$

The equation of any straight line through (2, 3)

$$y-3=m(x-2)$$

Now this line is perpendicular to the straight line $y=-x+11$ Hence from the condition of perpendicularity $m(-1)=-1$ giving $m=1$.

Hence the equation of the straight line through (2, 3) perpendicular to line $x+y-11=0$ is

$$y-3=(x-2)$$
$$y-x=1\cdots\cdots(2)$$
$$x+y=11\cdots\cdots(1)$$

Solving (1), (2) $2y=12$, $y=6$ $\therefore$ $x=5$

$\therefore$ Foot of perpendicular is (5, 6)

Ex. 5. Find the equation to the straight line passing through the intersection of lines $2x-3y+4=0$ and $3x+4y-5=0$ and perpendicular to the straight line $6x-7y+6=0$. [C. U.]

The equation to the line passing through the point of intersection of lines $2x-3y+4=0$ and $3x+4y-5=0$ is $2x-3y+4+k(3x+4y-5)=0$

$$\text{or,}\quad x(2+3k)+y(4k-3)+(4-5k)=0\cdots\cdots\text{(ii)}$$

This is perpendicular to $6x-7y+6=0\cdots\cdots$(i)

The condition of perpendicularity of lines (i) and (ii)

$$6(2+3k)-7(4k-3)=0$$

$$-10k+33=0 \quad \therefore \; k=\tfrac{33}{10}$$

$\therefore$ The required equation, on putting $k=\frac{33}{10}$

$$\tfrac{119}{10}x+\tfrac{102}{10}y-\tfrac{125}{10}=0 \quad \therefore \quad 119x+102y=125.$$

Exercises 3 (B)

1. Find the equation to the straight line, which passes through the point $(4, -5)$ and which is parallel to the straight line $3x+4y+3=0$.

2. Find the angle between the following pairs of st. lines
   (i) $y=3x+7$ and $3y=x+8$
   (ii) $x-4y=3$ and $6x-y=1$
   (iii) $x \cos 25^\circ+y \sin 25^\circ-7=0$
   and $x \sin 25^\circ-y \cos 25^\circ+7=0$

3. (i) Find the equation to the straight line through the point $(-4, 7)$ and perpendicular to the straight line $5x-7y+2=0$.

   (ii) Find the equation to the straight line which passes through (x', y') and perpendicular to the straight line
   $$yy'=2a(x+x').$$

4. Find the equation to the straight line which passes through (x_1, y_1) and is perpendicular to the join of (x_2, y_2) and (x_3, y_3). [C. U.]

5. Find the equation to the straight line which is at right angles to the line $\frac{x}{a}-\frac{y}{b}=1$ through the point where it meets the axis of x.

6. Find the co-ordinates of the points of intersection of the following straight lines

(i) $y=m_1x+\frac{a}{m_1}$ and $y=m_2x+\frac{a}{m_2}$

(ii) $\frac{x}{a}+\frac{y}{b}=1$ and $\frac{x}{b}+\frac{x}{a}=1.$ [C. U.]

7. Show that the lines

$$(a+b)x+(a-b)y-2ab=0$$

$$(a-b)x+(a+b)y-2ab=0 \text{ and } x+y=0$$

form an issosceles triangle whose vertical angle $2\tan^{-1}\frac{a}{b}$. [C.U.]

8. Find the foot of perpendicular

(i) from the origin to the straight line $y+x=1$

(ii) from the point (α, β) to the straight line $ax+by+c=0$

9. Find the equation to the straight line passing through the point (3, 2) and the point of intersection of $2x+3y=1$ and $3x-4y=6$.

10. Find the equation to the straight line through the point of intersection of lines $x+2y+3=0$ and $3x+4y+7=0$ and perpendicular to the straight line $y-x=8$.

11. Find the equation of the straight line passing through the intersection of the lines $x-2y-a=0$ and $x+3y-2a=0$ and parallel to the line $3x+4y=0$.

12. Find the equation to the straight line passing through intersection of the lines $3x-4y+1=0$ and $5x+y-1=0$ and cutting off equal intercepts from the axes.

13. Prove that three straight lines whose equations are $15x-18y+1=0$, $12x+10y-3=0$ and $6x+66y-11=0$ meet at a point.

Show also that the third line bisects the angle between the other two.

14. For all values of the parameter λ, prove that the line $5x+y-11-\lambda(2x-7y+3)=0$ goes through a fixed point and determine its co-ordinates. [C. U. 1950]

15. Find the equation to the straight line through the point (3, 2) and intersection of lines $3x+y-5=0$ and $x+5y+3$. Find also the area of the triangle cut off from the co-ordinate axes by the line. [C. U.]

16. Find the equation to the line through (3, 4) inclined at $\angle 45^\circ$ to the straight line $x-y=2$.

III-16. Length of perpendicular from a point upon a straight line

(I) Let the equation to the straight line AB, be $x \cos \alpha + y \sin \alpha = p$ ...(1)

The form of this equation suggests that length of perpendicular from the origin is p and α is the angle which the perpendicular makes with x-axis.

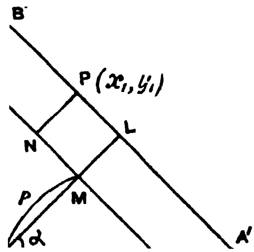


Fig. 17

Let P (x_1, y_1) be the given point from which the perpendicular is to be drawn upon the straight line AB.

Let the equation of straight line through P and parallel to the line AB be

$x \cos \alpha + y \sin \alpha = p'$ ...(2) where $p' = OL$

As this line passes through (x_1, y_1)

$$x_1 \cos \alpha + y_1 \sin \alpha = p'$$

$\therefore$ PN = Length of perpendicular from $x_1 y_1$ upon line (1)

$$= OL - OM = p' - p$$

$$= x_1 \cos \alpha + y_1 \sin \alpha - p \quad \cdots \quad \cdots \quad \text{(A)}$$

(II) Let the equation of the line be $ax + by + c = 0$ ...(3)

Comparing the equations (1) and (3)

$$\frac{-p}{c} = \frac{\cos \alpha}{a} = \frac{\sin \alpha}{b} = \pm \frac{\sqrt{\cos^2 \alpha + \sin^2 \alpha}}{\sqrt{a^2+b^2}} = \pm \frac{1}{\sqrt{a^2+b^2}}$$

whence, $\cos \alpha = \pm \frac{a}{\sqrt{a^2+b^2}}$, $\sin \alpha = \frac{\pm b}{\sqrt{a^2+b^2}}$, $-p = \frac{\pm c}{\sqrt{a^2+b^2}}$

Length of perpendicular upon the line $ax + by + c = 0$

$$\overline{a^2+b^2} \quad \sqrt{a^2+b^2}\rfloor$$

$$= \pm \frac{ax_1 + by_1 + c}{\sqrt{a^2+b^2}} \qquad \text{(B)}$$

Cor I. If c, the constant of the equation $ax+by+c=0$ be positive, then length of perpendicular from the origin $\dfrac{c}{\sqrt{a^2+b^2}}$.

Cor II. It is seen that the length of the perpendicular from a point P to a straight line AB may be positive or negative. When the absolute length is required, that sign is to be taken which makes the expression positive. But when the directed distance is required, due regard should be paid. The straight line AB divides the entire plane into two portions and any point may lie in any part. If the point P and the origin O lie on the same side of the straight line, both the perpendiculars are drawn in the same direction and hence the same sign is to be taken. But if P and O lie on opposite sides, the perpendicular from P must have the sign opposite to that from O.

III-17. The equation to the lines besecting the angle between two straight lines.

Let BAB' and CAC' be two straight lines whose equations are $a_1x+b_1y+c_1=0$ ... (1)

and $a_2x+b_2y+c_2=0$ ... (2)

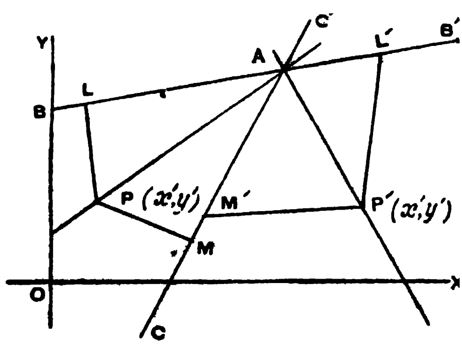


Fig. 18

Let P (x', y') be a point on the bisector of angle BAC between two straight lines. Take both c_1, c_2 positive.

Let PL, PM be perpendiculars from P upon two straight lines (1) and (2). Then we see that these perpendiculars PL, PM and perpendiculars from O upon these lines are in the same sense and hence are of same sign. Now the lengths of perpendicular from $P(x'y')$ upon (1) and (2) are equal. So that $\frac{a_1x'+b_1y'+c_1}{\sqrt{a_1^2+b_1^2}}=\frac{a_2x'+b_2y'+c_2}{\sqrt{a_2^2+b_2^2}}$

If we take P' on the bisector of $\angle CAB'$, we see that perpendicular $P'L'$ and the perpendicular from O upon BAB' are of same sense but the perpendicular $P'M'$ and the perpendicular from O upon AC are of opposite sense. Hence the perpendiculars $P'L'$, $P'M'$ are equal in magnitude but opposite in sign.

So that $\frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}}=-\frac{a_2x+b_2y+c_2}{\sqrt{a_2^2+b_2^2}}$

Making $x'y'$ current co-ordinates, we get the equation of bisectors of angles

$$\frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}}=\pm\frac{a_2x+b_2y+c_2}{\sqrt{a_2^2+b_2^2}}$$

The positive sign refers to the case when the bisector lies in the origin side or non-origin side of both the lines and the negative sign refers to the case when the bisector lies in the angle in which the origin lies on the positive side of one and non-origin side of the other.

ILLUSTRATIVE EXAMPLES

Ex. 1 To find the length of perpendicular from the point $(-3, -4)$ upon the straight line $12(x+6)=5(y-2)$.

The equation of the line

$$12x+72=5y-10$$

$$12x-5y+82=0$$

The length of perpendicular from $(-3, -4)$ upon the line $=\dfrac{12(-3)-5(-4)+82}{\sqrt{12^2+(-5)^2}}$

$$=\frac{-36+20+82}{\sqrt{169}}=\frac{66}{13}=5\frac{1}{13}$$

Ex. 2. Find the distance between two parallel straight lines $ax+by+c=0$, $ax+by+d=0$

Let p_1 = Length of perpendicular from the origin $(0, 0)$ to the straight line $ax+by+c=0$

$$=\frac{a\times 0+b\times 0+c}{\sqrt{a^2+b^2}}=\frac{c}{\sqrt{a^2+b^2}}$$

p_2 = Length of perpendicular from the origin $(0, 0)$ to the straight line $ax+by+d=0$

$$=\frac{a\times 0+b\times 0+d}{\sqrt{a^2+b^2}}=\frac{d}{\sqrt{a^2+b^2}}$$

Distance between two parallel lines

$$=p_1-p_2=\frac{c-d}{\sqrt{a^2+b^2}}$$

Ex. 3. Find the equations to the bisectors of the angles between the straight lines

$$3x-4y+7=0 \text{ and } 12x-5y-8=0.$$

Writing the equation so that their constant terms are positive

$$3x-4y+7=0 \text{ and } -12x+5y+8=0$$

$\therefore$ The equation of bisector of angle in which the origin lies

$$\frac{3x-4y+7}{\sqrt{3^2+4^2}}=\frac{-12x+5y+8}{\sqrt{(-12)^2+5^2}}$$

$$\frac{3x-4y+7}{5}=\frac{-12x+5y+8}{13}$$

$$99x-77y+51=0.$$

The equation of bisector ot angle in which the origin does not lie

$$\frac{3x-4y+7}{5}=-\frac{12x+5y+8}{13}$$

$$21x+27y-131=0$$

Exercises 3(C)

1. Find the equations to the straight lines which
(i) bisects the angle in which the origin lies
and (ii) bisects the angle in which the origin does not lie.
(a) $4x+3y-7=0$ and $24x+7y-31=0$
(b) $2x+y=4$ and $y+3x=5$

2. Find the length of perpendicular drawn from the point (4, 5) upon the straight line $3x+4y=10$.

3. If p and p' be the perpendiculars from the origin upon the straight lines $x \sec\theta+y \operatorname{cosec}\theta=a$ and $x\cos\theta-y\sin\theta=a\cos 2\theta$.
prove that $4p^2+p'^2=a^2$.

4. If p and p' be the perpendiculars from the origin upon lines

$$x\sin\theta+y\cos\theta=\frac{a}{2}\sin 2\theta$$

$$x\cos\theta-y\sin\theta=a\cos 2\theta$$

prove that $4p^2+p'^2=a^2$. [C. U.]

5. Find the distance between two parallel lines

$$y=mx+c$$

$$y=mx+d.$$

6. What are the points on the axis of x whose perpendicular distance from the straight line $\frac{x}{a}+\frac{y}{b}=1$ is a

Writing the equation in the form $bx+ay-ab=0$...... (1)
Let us take the point $(k, 0)$ on x-axis.
a=length of perpendicular from $(k, 0)$ upon (1)

$$a=\pm\frac{bk-ab}{\sqrt{a^2+b^2}} \quad \therefore \quad a\sqrt{a^2+b^2}=\pm bk-ab$$

$\therefore \quad \pm bk = ab + a\sqrt{a^2+b^2}$ whence

$$k = \pm \frac{a[b+\sqrt{a^2+b^2}]}{b} \qquad \ldots\ldots(2)$$

The required points are $(k, 0)$ where k is given by (2).

7. How far the line $h(x+h)+k(y+k)=0$ from the origin ?

8. Find the distance between two parallel lines

$3x+4y-15=0$ and $6x+8y+15=0$.

9. Show that the perpendiculars let fall from any point of the straight line $7x-9y+10=0$ upon two straight lines $3x+4y=5$ and $12x+5y=7$ are eqtual to each other.

10. Show that the perpendiculars let fall from any point of the straight line $2x+11y=5$ upon two straight lines $24x+7y=20$ and $4x-3y=2$ are equal to each other.

11. If the sum of the perpendiculars dropped from a variable point P on the two lines $x+y-5=0$ and $3x-2y+7=0$ be equal *to 10*, prove that P must move on a right line. [C. U. 1950]

12. A point P moves so that the perpendiculars from it to the lines $x+2y-1=0$ and $4x-2y=3$ are equal. Find the locus of P.

Transformation of Co-ordinates

III-18. To change the origin of co-ordinates without changing the direction of axes.

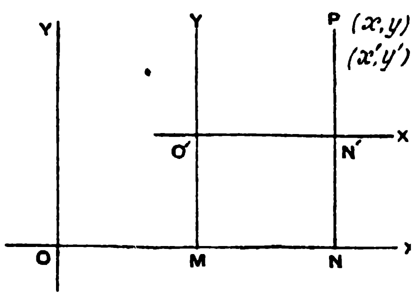


Fig. 19

Let OX , and OY be the original axes and let the new axes parallel to OX, OY respectively be $O'X'$, $O'Y'$.

Let O' be new origin whose co-ordinates referred to OX, OY be (h, k)

Let (x, y) be co-ordinates of the point P referred to original axes and (x', y') be the co-ordinates of P referred to new axes.

Draw PN perpendicular from P upon OX, and let PN cut OX' in N'. Then $ON=x$, $NP=y$, $O'N'=x'$, $N'P=y'$, $OM=h$, $MO'=k$

$\therefore$ We have $x=ON=OM+MN=x'+h$

$$y=NP=NN'+N'P$$
$$=MO'+N'P=y'+k.$$

Hence if the origin be transferred to $O'(h, k)$ we are to substitute for the co-ordinates x and y the quantities $x'+h$ and $y'+k$

ILLUSTRATIVE EXAMPLES

Ex. 1. Transform to parallel axes through the point $(1, -2)$ the equation $x^2-4x+4y+8=0$

As the new origin is $(1, -2)$ the formula for transformation
$$x=1+x'$$
$$y=-2+y'$$

Substituting the values of x in the equation
$$(x'+1)^2-4(x'+1)+4(y'-2)+8=0$$
$$x'^2+2x'+1-4x'-4+4y'-8+8=0$$

Make the co-ordinates $(x'\ y')$ current co-ordinates (x, y)
$$x^2-2x+4y-3=0$$

Exercises 3 (D)

1 Transform to parallel axes through the point $(-2, 3)$ the equation $2x^2+4xy+3y^2-4x-22y+7=0$.

2. Transform to parallel axes through the point $(1, -2)$ the equation (1) $y^2-4x+4y+8=0$

and (2) $2x^2+y^2-4x+4y=0$

3. By transforming to parallel axes through a properly chosen point (h, k) prove the equation $12x^2-10xy+2y^2+11x-5y+2=0$ can be reduced to one containing only terms of the second degree. [C. U.]

CHAPTER IV

CIRCLE

IV-1. A circle is the locus of a point which moves such that its distance from a fixed point is constant.

The fixed point is called the centre and the constant distance is the radius of the circle.

IV-2. (*a*) To find the equation to a circle whose centre is the origin and whose radius is equal to r.

Let $P(x, y)$ be the moving point of the circle.

Draw PN perpendicular upon the x-axis, then $PN=y$, $ON=x$.

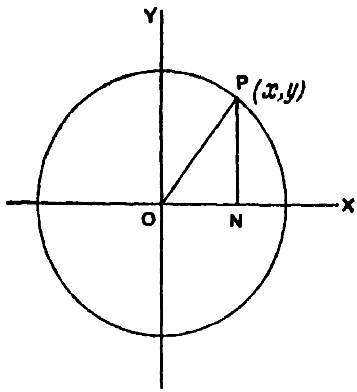


Fig. 20

Now ON^2+NP^2

$=OP^2=r^2=\text{constant}$

$$x^2+y^2=r^2$$

(*b*) When the centre is the point (h, k)

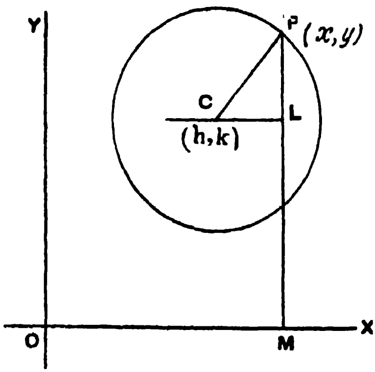


Fig. 21

Let C (h, k) be centre of the circle and r be the radius of the circle. Let $P(x, y)$ be moving point, then

$$CP^2=\text{constant}=r^2.$$

Now CP is the distance from $P(x, y)$ to $C(h, k)$

$\therefore\ CP^2=(x-h)^2+(y-k)^2$

$\therefore$ The equation of circle becomes

$$(x-h)^2+(y-k)^2=r^2$$

IV-3. The nature of the locus whose equation is given to be $x^2+y^2+2gx+2fy+c=0$.........(1)

The equation can be put in the form

$$(x^2+2gx+g^2)+(y^2+2fy+f^2)=g^2+f^2-c$$

$$(x+g)^2+(y+f)^2=(\sqrt{g^2+f^2-c})^2$$

$$\{x-(-g)\}^2+\{y-(-f)\}^2=(\sqrt{g^2+f^2-c})^2$$

$\therefore$ The point moves in such a way that the distance of the moving point (x, y) from the point $(-g, -f)$ is constant and equal to $\sqrt{g^2+f^2-c}$.

$\therefore$ The locus of P is a circle whose centre is $(-g, -f)$ and radius equal to $\sqrt{g^2+f^2-c}$

N. B. If $g^2+f^2>c$, the circle is real

and $g^2+f^2<c$, the circle is imaginary

and if $g^2+f^2=c$, the locus reduces to a point circle at $(-g, -f)$.

IV-4. The condition that the general equation of the second degree $ax^2+by^2+2hxy+2g_1x+2f_1y+c_1=0$ represent a circle ......(2)

Multiplying the equation (1) by a, we have

$$ax^2+ay^2+2gax+2fay+ca=0 \cdots\cdots(3)$$

Compairing the equations (2) & (3) we see that

$$a=b, \quad h=0$$

$\therefore$ The condition of the circle

$a=$coefficient of $x^2=b=$coefficient of y^2

and $h=$coefficient of $xy=0$.

ILLUSTRATIVE EXAMPLES

Ex. 1. Find the centre and radius of the circle

$$5x^2+5y^2-4x-3y=15$$

Dividing the equation by 5, we have

$$x^2+y^2-\frac{4}{5}x-\frac{3}{5}y=3$$

$$\text{or,}\quad \left(x^2-\frac{4}{5}x+\frac{4}{25}\right)+\left(y^2-\frac{3}{5}y+\frac{9}{100}\right)=3+\frac{4}{25}+\frac{9}{100}$$

$$\left(x-\frac{2}{5}\right)^2+\left(y-\frac{3}{10}\right)^2=\frac{300+16+9}{100}=\frac{\overset{13}{\cancel{325}}}{\cancel{100}}$$

$$\therefore\quad \left(x-\frac{2}{5}\right)^2+\left(y-\frac{3}{10}\right)^2=\left(\frac{\sqrt{13}}{2}\right)^2$$

Center is $\left(\frac{2}{5}, \frac{3}{10}\right)$ and radius – $\frac{\sqrt{13}}{2}$

Ex. 2. To find the equation to the circle whose diameter is the join of $(x_1\ y_1)$ $(x_2\ y_2)$.

Let $B\ (x_1, y_1)$, $C\ (x_2\ y_2)$ be the given points and $P(x'\ y')$ be any point on the circle.

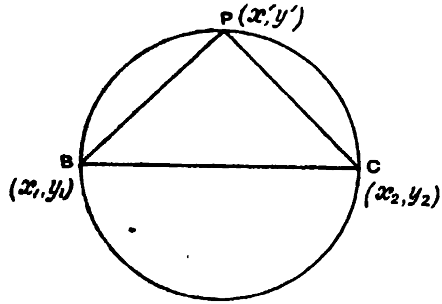


Fig. 22

Then $\angle BPC$ is a right angle (angle in semicircle is equal to 90°.) So that BP, CP are perpendiculars.

Now the equation of BP, CP

$$y-y_1=\frac{y'-y_1}{x'-x_1}\ (x-x_1)$$

$$\text{and}\quad y-y_2=\frac{y'-y_2}{x'-x_2}\ (x-x_2)$$

The condition that these two straight lines are at right angles is $\frac{y'-y_1}{x'-x_1}\cdot\frac{y'-y_2}{x'-x_2}=-1$

Making (x', y') current co-ordinates (x, y) we get the locus of $P(x, y)$

$$(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0$$

Ex. 3. To find the length of intercepts on axes of the circle represented by the equation

$$x^2+y^2+2gx+2fy+c=0 \quad \cdots\cdots\text{(i)}$$

This circle cuts the x-axis at two points given by putting $y=0$ (*viz.* at A, B)

$$x^2+2gx+c=0 \cdots \quad \cdot\text{(ii)}$$

If two points be

$$(x_1\ 0)\ (x_2\ 0_2)$$

$x_1+x_2=$ sum of roots of equation (ii)

$$=-2g$$

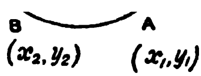


Fig. 28

$x_1\ x_2=$ product of roots $=c$

$$(x_1-x_2)^2$$

$$=(x_1+x_2)^2-4x_1x_2=4g^2-4c$$

$$AB=x_1-x_2=2\sqrt{g^2-c}.$$

Similarly when $x=0$, equation (i) reduces to

$$y^2+2fy+c=0$$

$$y_1+y_2=-2f, \quad y_1\ y_2=c$$

$$(y_1-y_2)^2=(y_1+y_2)^2-4y_1\ y_2$$

$$=4f^2-4c=4(f^2-c)$$

Intercept on y-axis $=y_1-y_2=2\sqrt{f^2-c}$.

N. B. If the circle touches both axes, these values of intercepts are zero. So that $c=g^2=f^2=\lambda^2$ say.

In this case, the circle touches the axes

$\therefore$ The equation of the circle touching the axes can be put in the form $x^2+y^2\pm2\lambda x\pm2\lambda y+\lambda^2=0$

Ex. 4. To find the equation to circle which passes through (1, 2), (3, −4) and (5, −6).

Let the equation to circle $x^2+y^2+2gx+2fy+c=0$

The circle passes through (1, 2)

$$\therefore \quad 1+4+2g+4f+c=0 \quad \cdots\cdots\cdots(1)$$

The circle passes through (3, −4)

$$\therefore \quad 9+16+6g-8f+c=0 \cdots\cdots\cdots(2)$$

The circle passes through $(5, -6)=0$

$$25+36+10g-12f+c=0 \cdots\cdots(3)$$

From (1) and (2) $4g-12f+20=0 \quad \cdots\cdots\cdots(4)$

From (2) and (3) $4g-4f+36=0 \quad \cdots\cdots\cdots(5)$

From (4) and (5) $-8f-16=0$

$$\therefore \quad 2f=-4 \qquad 2g=-22$$

$$c=25$$

Hence the equation of the circle

$$x^2+y^2-22x-4y+25=0.$$

Ex. 5. To find the equation to the circle which touches each axis at a distance 5 from the origin.

Since the perpendicular lines to x-axis and y-axis at the points of contact pass through the centre, the centre is clearly on $y=5$ and $x=5$. So that centre is (5, 5)

The radius of circle $=5$

Hence the equation to the circle

$$(x-5)^2+(y-5)^2=5^2$$

$$\therefore \quad x^2+y^2-10x-10y+25=0$$

IV-5. The *points of intersection of a straight line and the circle : Condition of Tangency*

Let the equation of circle be $x^2+y^2=a^2 \quad \cdots\cdots\cdots(1)$

and the equation of the straight line $y=mx+c \quad \cdots\cdots(2)$

The co-ordinates of the points of intersection satisfy both equation (1) & (2) and hence x-co-ordinates of the points of intersection satisfy the equation

$$x^2+(mx+c)^2=a^2$$

or, $x^2(1+m^2)+2mcx+(c^2-a^2)=0\cdots\cdots\cdots(3)$

This is a quadratic equation in x, giving two values of x which are x-co-ordinates of two points of intersection of (1) and (2).

If two points of intersection coincide, the straight line touches the circle. In this case, for equal roots of equation (3), the discrimant of the equation (3) is zero.

$$\therefore \quad 4m^2c^2=4(1+m^2)(c^2-a^2)$$

$$\text{or,} \quad m^2c^2=c^2+m^2c^2-a^2-a^2m^2$$

giving $c^2=a^2(1+m^2)$ which is the condition of tangency of line (1)

$$\therefore \quad c=\pm a\sqrt{1+m^2}$$

Hence $y=mx\pm a\sqrt{1+m^2}$ touches the circle for all values of m.

IV-6. To find the equation of the tangent to the circle at $(x_1\ y_1)$

(i) Let the equation of circle $x^2+y^2=a^2$

Let $(x_1\ y_1)\ (x_2\ y_2)$ be any two points on the circle.

So that $x_1^2+y_1^2=a^2$

$$x_2^2+y_2^2=a^2$$

When $x_1^2-x_2^2+y_1^2-y_2^2=0$

$$(x_1-x_2)(x_1+x_2)=-(y_1-y_2)(y_1+y_2)$$

$$\therefore \quad \frac{y_1-y_2}{x_1-x_2}=-\frac{x_1+x_2}{y_1+y_2} \quad \ldots\ldots\ldots(1)$$

Now the equation to the straight line joining $(x_1, y_1$, $(x_2\ y_2)$

$$y-y_1=\frac{y_1-y_2}{x_1-x_2}(x-x_1)$$

$$\therefore \quad \text{From (1) } y-y_1=-\frac{x_1+x_2}{y_1+y_2}(x-x_1) \quad \ldots\ldots\ldots(2)$$

If point (x_2, y_2) approaches the point (x_1, y_1) and ultimately coincides with the point, we get the chord to be tangent at $(x_1\ y_1)$. In that case $x_2 = x_1$ and $y_2 = y_1$

Now equation (2) becomes

$$y - y_1 = -\frac{2x_1}{2y_1}(x - x_1)$$

$$yy_1 - y_1^2 = -xx_1 + x_1^2$$

$\therefore$ Equation of tangent at $(x_1\ y_1)$

$$xx_1 + yy_1 = x_1^2 + y_1^2 - a^2.$$

(ii) Let the equation of the circle be

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

Let (x_1, y_1), (x_2, y_2) be any two points on the circle.

Then $x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0$

$$x_2^2 + y_2^2 + 2gx_2 + 2fy_2 + c = 0$$

Therefore by subtraction

$$(x_2^2 - x_1^2) + 2g(x_2 - x_1) + (y_2^2 - y_1^2) + 2f(y_2 - y_1) = 0$$

$$(x_2 - x_1)(x_2 + x_1 + 2g) + (y_2 - y_1)(y_2 + y_1 + 2f) = 0$$

$$\frac{y_2 - y_1}{x_2 - x_1} = -\frac{x_2 + x_1 + 2g}{y_2 + y_1 + 2f}$$

Now the equation of the straight line joining $(x_1\ y_1)$ and $(x_2\ y_2)$

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

or, $$y - y_1 = -\frac{x_2 + x_1 + 2g}{y_2 + y_1 + 2f}(x - x_1) \quad \cdots\cdots(A)$$

If the point (x_2, y_2) approaches and ultimately coincides with $(x_1\ y_1)$, we get the limiting position of chord and hence the tangent at (x_1, y_1).

In that case the equation (A) transforms to

$$y - y_1 = -\frac{2(x_1 + g)}{2(y_1 + f)}(x - x_1)$$

$$yy_1+yf-y_1^2-fy_1=-xx_1-gx+x_1^2+gx_1$$

or, $xx_1+yy_1+gx+fy=x_1^2+y_1^2+gx_1+fy_1$

Adding gx_1+fy_1+c to both sides, we get

$$xx_1+yy_1+g(x+x_1)+f(y+y_1)+c$$
$$=x_1^2+y_1^2+2gx_1+2fy_1+c=0$$

$\therefore$ Equation of tangent at (x_1, y_1)

$$xx_1+yy_1+g(x+x_1)+f(y+y_1)+c=0$$

IV-7. To find the equation of normal at $(x_1\ y_1)$ to

(I) circle, $x^2+y^2=a^2$

(II) circle, $x^2+y^2+2gx+2fy+c=0.$

(I) The equation of the tangent to the circle

$x^2+y^2=a^2$ at (x_1, y_1) is

$$xx_1+yy_1=a^2$$

or, $y=-\frac{x_1}{y_1}x+\frac{a^2}{y_1}$ ... ... (i)

So that m of the tangent $=-\frac{x_1}{y_1}$

Now the equation of any straight line through the point $(x_1\ y_1)$ is $y-y_1=m'(x-x_1)$ ... ... (ii)

The condition of perpendicularity of (i) and (ii)

$$mm'=-1 \text{ or, } \left(-\frac{x_1}{y_1}\right)m'=-1$$

$$m'=\frac{y_1}{x_1}$$

$$y-y_1=\frac{y_1}{x_1}(x-x_1)$$

$$yx_1-x_1y_1=xy_1-x_1y_1$$

$\therefore$ The required equation of normal at $x_1\ y_1$,

$$yx_1-xy_1=0$$

(II) The equation of the tangent to the circle
$x^2+y^2+2gx+2fy+c=0$ at (x_1, y_1)

$$xx_1+yy_1+g(x+x_1)+f(y+y_1)+c=0$$

$$y(y_1+f)+x(x_1+g)+(gx_1+fy_1+c)=0$$

$$y=-\frac{x_1+g}{y_1+f}x-\frac{gx_1+fy_1+c}{y_1+f} \quad \cdots \quad \cdots \quad \text{(i)}$$

So that m of tangent $=-\dfrac{x_1+g}{y_1+f}$

Let the equation of normal be

$$y-y_1=m'(x-x_1) \quad \cdots \quad \cdots \quad \text{(ii)}$$

The condition of perpendicularity of lines (i) and (ii) is

$$m'\left(-\frac{x_1+g}{y_1+f}\right)=-1$$

$$\text{whence } m'=\frac{y_1+f}{x_1+g}$$

The equation of normal at $(x_1\ y_1)$

$$y-y_1=\frac{y_1+f}{x_1+g}(x-x_1)$$

or, $\dfrac{y-y_1}{y_1+f}=\dfrac{x-x_1}{x_1+g}$

or, $x(y_1+f)-y(x_1+g)-fx_1+gy_1=0.$

IV-8. Chord having its middle point given.

Let the middle point N of chord AB be $(x_1\ y_1)$.

Then ON is perpendicular to the chord AB.

Let the equation of chord be

$$y-y_1=m(x-x_1) \quad \cdots \quad (1)$$

Now the equation of ON

$$y=\frac{y_1}{x_1}\cdot x \quad \cdots \quad (2)$$

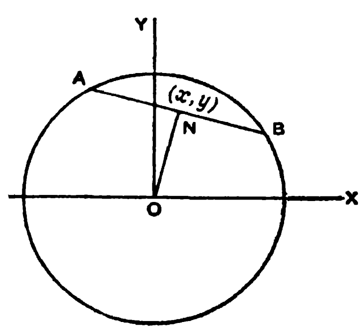


Fig. 24

The condition of perpendicularity of lines (1) and (2)

$$m.\ \frac{y_1}{x_1}=-1 \text{ giving } m=-\frac{x_1}{y_1}.$$

Hence the equation of the chord having $x_1\ y_1$ as its middle point $y-y_1=-\frac{x_1}{y_1}(x-x_1)$

$$i.e.,\quad (x-x_1)\,x_1+(y-y_1)\,y_1=0$$

ILLUSTRATIVE EXAMPLES

Ex. 1. Find the equations of the tangent to the circle $x^2+y^2=9$ which are parallel to $3x+4y=0$ [C. U]

Equation of any tangent can be written as

$$y=mx\pm 3\sqrt{1+m^2} \quad \cdots \quad \cdots \quad (1)$$

This is parallel to $3x+4y=0$

$$i.e.,\quad y=-\frac{3}{4}x \quad \cdots \quad \cdots \quad (2)$$

The condition that the lines (1) and (2) are parallel is

$$m=-\frac{3}{4}$$

∴ The equation of the tangent becomes

$$y=-\frac{3}{4}x\pm 3\sqrt{1+\frac{9}{16}}$$

$$y=-\frac{3}{4}x.\pm 3.\frac{5}{4}$$

$$4y+3x=\pm 15.$$

Ex. 2. Find the equations of two tangents to the circle $x^2+y^2=3$ which makes angle 60° with the axis of x.

[C. U. 1955]

Compairing the equation with $x^2+y^2=a^2$, we get $a=\sqrt{3}$. Now equation of any tangent to the circle can be written as

$$y=mx\pm\sqrt{3}\sqrt{1+m^2}$$

Here $m=\tan 60^\circ=\sqrt{3}$

∴ The equation of the tangent $y=\sqrt{3}\,x\pm\sqrt{3}\sqrt{1+3}$

$$y=\sqrt{3}\,x\pm 2\sqrt{3}$$

Ex. 3. Find the equation of the tangent and the normal to the circle $x^2+y^2-14x+8y+40=0$, at the point $(3,-1)$.

Compairing this equation with the standard equation

$x^2+y^2+2gx+2fy+c=0$, we get $g=-7$, $f=4$ and $c=4$.

The equation of tangent at (x_1, y_1) is

$$xx_1+yy_1+g\,(x+x_1)+f\,(y+y_1)+c=0$$

This becomes

$$x\,.\,3+y\,(-1)-7\,(x+3)+4\,(y-1)+40$$

On simplification, $-4x+3y=-15$

$$i.e., \quad 4x-3y=15$$

Also the equation of normal at (x_1, y_1) is

$$x\,(y_1+f)-y\,(x_1+g)-fx_1+gy_1=0$$

Here $f=4$, $g=-7$.

The equation of normal at $(3, -1)$ becomes

$$x\,(-1+4)-y\,(3-7)-4\,.\,3-7\,(-1)=0$$

$$i.e., \quad 3x+4y=5.$$

Ex. 4. Find the equations of tangents to the circle $x^2+y^2-6x+4y=7$ which is perpendicular to the straight line $y=2x+3$

The equation of the circle can be put in the form

$$(x^2-6x+9)+(y^2+4y+4)=7+13$$

$$(x-3)^2+(y+2)^2=20$$

$\therefore$ The centre of this circle $(3, -2)$ and radius $=\sqrt{20}$
$=2\sqrt{5}$.

The equation to the straight line which is perpendicular to $y=2x+3$ can be written as

$$y=-\frac{1}{2}x-c \qquad \left[\because\ 2\times\left(-\tfrac{1}{2}\right)=-1\right]$$

$$x+2y+2c=0 \quad \cdots \quad \cdots \quad \cdots \quad (1)$$

If this straight line (1) is tangent to the circle, the length of perpendicular from the centre $(3, -2)$ upon the tangent (1) will be equal to the radius $2\sqrt{5}$.

Now the length of the perpendicular from (3, −2) upon $x+2y+2c=0$ is

$$\frac{3+2(-2)+2c}{\sqrt{1+4}}=\pm 2\sqrt{5}$$

$$\frac{2c-1}{\sqrt{5}}=\pm 2\sqrt{5},\qquad 2c-1=\pm 2\sqrt{5}\sqrt{5}$$

$$2c-1=\pm 10$$

$$2c=\pm 11$$

∴ The equation of tangent is $x+2y\pm 11=0$.

Ex. 5 Find the co-ordinates of the point of contact of the tangent $y=mx+a\sqrt{1+m^2}$ to the circle $x^2+y^2=a^2$.

Let $(x_1\ y_1)$ be the point of contact

Then we get the equation of the tangent at $(x_1\ y_1)$

$$xx_1+yy_1=a^2$$

Comparing it with the equation given

$$-mx+y=a\sqrt{1+m^2}$$

$$\frac{x_1}{-m}=\frac{y_1}{1}=\frac{a^2}{a\sqrt{1+m^2}}$$

giving $x_1=\dfrac{-am}{\sqrt{1+m^2}}$ and $y_1=\dfrac{}{\sqrt{1+m^2}}$.

Ex 6. Find the length of tangent from $(x_1\ y_1)$ to the circle $x^2+y^2+2gx+2fy+c=0$

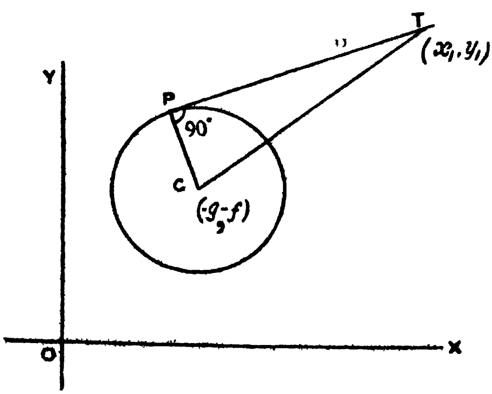


Fig. 25

Let T be the point $(x_1\ y_1)$ and P be the point of contact

The centre of the circle is $C(-g, -f)$ and the radius of this circle is $\sqrt{g^2+f^2-c}$

[Length of tangent from $(x_1\ y_1)$]$^2 = PT^2 = CT^2 - CP^2$

$$=(x_1+g)^2+(y_1+f)^2-(\sqrt{g^2+f^2-c})^2$$
$$=x_1^2+2gx_1+g^2+y_1^2+2fy_1+f^2-(g^2+f^2-c)$$
$$=x_1^2+y_1^2+2gx_1+2fy_1+c$$

$=\sqrt{s_1}$. Where s_1 denotes value of the expression in x, y in the equation when x_1, y_1 are subtituted for x, y.

Exercises 4

1. Find the centre and radius of the following circles :

(a) $x^2+y^2-4x-8y=41$
(b) $x^2+y^2=4(x+y)$
(c) $5x^2+5y^2=2x+3y$
(d) $x^2+y^2=2gx-2fy$
(e) $x^2+y^2+6x-8y=24$.

2. Find the equation to the circle passing through the points

(a) (5, 7), (8, 1) and (1, 3)
(b) (a, b), $(a, -b)$ and $(a+b, a-b)$
(c) (0, 0), (0, −6) and (3, 4)
(d) (1, 2), (−3, 4) and (5, −6).

3. $ABCD$ is a square of side a, taking AB and AD as axes, prove that the equation to the circle circumscribing the square is $x^2+y^2=a(x+y)$. [C. U. 1951]

4. Find the equation to the circle which passes through the origin and cut off intercepts equal to 3 and 4 from the axes.

5. Find the equation to the circle which touches the axes and of radius a.

6. Find the equation to the circle which touches the axis of x at the origin and passes through the point (b, c).

7. Find the equation to the circle having the join of (2, 3) and (5, 6) as diameter.

8. Prove that the centres of three circles $x^2+y^2=1$, $x^2+y^2+6x-2y=1$, $x^2+y^2-12x+4y=1$ lie on a right line. [C. U.]

9. Find the equation to the circle which touches the co-ordinate axes at (1, 0) and (0, 1). [C. U.]

10. (*a*) Prove that two circles $x^2+y^2+4x-10y-20=0$, $x^2+y^2-8x+6y-16=0$ touch each other externally.

[**Hint** :—The distance between centres of two circles = sum of their radii.]

(*b*) Prove that two circles $x^2+y^2+4x-6y-36=0$ and $x^2+y^2+16x+3y+22=0$ touch each other internally.

[**Hint** :—The distance between centres of two circles

= difference of their radii.]

11. Obtain the equation of the circle which passes through two points on the axis of x which are at a distance 2 from the origin and whose radius is 5.

12. Find the equation to the circle which touches the axis of y at a distance $+4$ from the origin and cuts off an intercept 6 from the axis of x.

13. Find the equation to the circle which goes through the origin and cuts off intercept equal to h and k from the positive sides of the axes.

14. Find the equation of circle concentric to the circle $x^2+y^2+8x-10y+16=0$ and passes through the point (1, 2).

15. Find the equation to the circle which is concentric with the circle and which passes through $(5, -2)$.

16. Write down the equation of tangent to

(*a*) the circle $x^2+y^2-3x+10y-15=0$ at $(4, -11)$

(*b*) the circle $4x^2+4y^2-16x+24y=117$ at $(-4, -\frac{11}{2})$.

17. Find the equation of tangent to the circle $x^2+y^2=4$ which is parallel to $x+2y+3=0$.

18. Find the equation of tangent to the circle $x^2+y^2-6x+4y=12$ which is parallel to the line $4x+3y+5=0$. [C. U.]

19. Prove that $x=7$ and $y=8$ both touch the circle $x^2+y^2-4x-6y=12$. Find the point of contact. [C. U. 1955]

20. If the tangents at (x_1y_1) and (x_2y_2) on the circle $x^2+y^2+2gx+2fy+c=0$ are perpendicular,
prove $x_1x_2+y_1y_2+g(x_1+x_2)+f(y_1+y_2)+g^2+f^2=0$.

21. Prove that the straight line $y=x+c\sqrt{2}$ touches the circle and find the point of contact. [C. U.]

22. Find the condition that the straight line $Ax+By+C=0$ may touch the circle $(x-a)^2+(y-b)^2=c^2$.

23. Find the equation to the tangent and normal at $(-5, 2)$ to the circle $x^2+y^2+3x-4y-6=0$.

24. The length of the tangent from (f, g) to the circle $x^2+y^2=6$ is twice the length of tangent to the circle $x^2+y^2+3x+3y=0$. Show that $g^2+f^2+4f+4g+2=0$.

25. Find the condition that the line $lx+my+n=0$ should be (i) a tangent (ii) a normal to the circle $x^2+y^2+2gx+2fy+c=0$.

26. Prove that if $y=x \sin \alpha+a \sec \alpha$ be a tangent to the circle $x^2+y^2=a^2$ then $a^2-\cos^2\alpha=1$.

CHAPTER V
PARABOLA

V-1. A parabola is the locus of a point which moves in a plane in such manner that its distance from a fixed point, called focus is always equal to its distance from a fixed line, called the directrix.

The parabola is a particular kind of curves called conic sections or conics. The definition of conics is as follows :

If a point moves in such a manner that its distance from a fixed point, called the focus bears a constant ratio to its distance from a fixed line, called the directrix. Then the moving point traces a curve called the conics or conic sections. If the constant ratio, generally denoted by 'e' $=1$, the curve is the parabola and if $e<1$, the curve is called the ellipse and if $e>1$, the curve is the hyperbola. The name 'conic section' comes from the fact that (i) if we intersect a double right circular cone by a plane perpendicular to the axis, the section becomes a circle, (ii) if we intersect the said cone by a plane in an oblique manner (not at right angle to the axis) the section becomes an ellipse, a parabola or a hyperbola according as the plane makes with the axis of the cone an angle $\gtreqless$ the semi-vertical angle. The section is, however, a pair of straight lines if the plane be drawn through the vertex of the cone.

V-2. The equation to the parabola : standard form.

Let S be the focus and MM' be the directix.

Draw $X'S$ perpendicular to MM'. Bisect SX' at O.

The line $X'OX$ is called the axis of the parabola.

Draw OY, perpendicular to OX. The point O is called the vertex of the parabola.

Take $X'OX$ as x-axis and OY as y-axis.

Let $P(x, y)$ be any point on the parabola. Draw PM perpendicular to MM' and join SP. Draw PN perpendicular

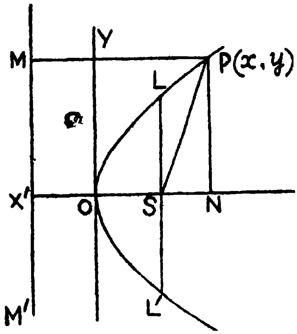


Fig. 26

to OX. Take $X'O = OS = a$. So S is the point (a, o). Then by definition of parabola $SP = PM = X'N$.

$$\therefore \quad SP = X'O + ON$$

$$SP^2 = (X'O + ON)^2$$

$$(x-a)^2 + (y-o)^2 = (a+x)^2$$

$$y^2 = (x+a)^2 - (x-a)^2$$

$y^2 = 4\,ax$ which is the standard equation of the parabola.

Cor. I. The equation of the directrix $x = -a$.

Cor. II. If we take LSL' perpendicular to the axis $X'X$, the line LSL' is called the latus rectum.

The latus rectum is defined as the focal chord, perpendicular to the axis.

Now x-co-ordinate of $L = a$ and if we put $x = a$,

$$y^2 = 4a\,.\,a = 4a^2 \qquad SL = y = 2a$$

$LL' = 2SL = 4a = 4$ (distance of S from O, the vertex).

Cor. III. For any value of x, we get two equal and opposite values of y. Hence any line parallel to y-axis, cut the parabola in two points whose y-co-ordinates are equal and opposite. Hence the curve is symmetrical with respect to x-axis.

V-3. Other form of the parabolas.

(i) If we take the equation of the directrix as $x = a$ and focus S', as $(-a, o)$ the equation of the parabola $y^2 = -4ax$. [fig. 27]

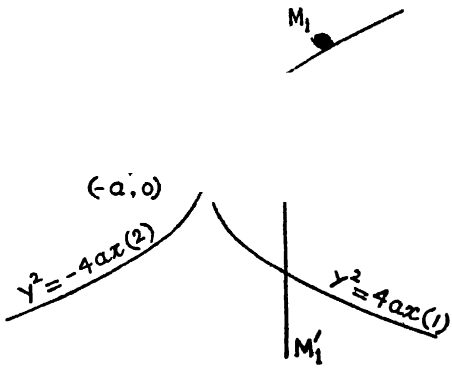


Fig. 27

(ii) If we take $x = +a$ (line M_1M_1') as directrix and focus S' $(-a, o)$.

The equation of the parabola $y^2 = -4ax$.

(ii) If we take $y = -a$ as equation of directrix and the point S (o, a) as the focus, the equation of the parabola becomes

$x^2 = 4ay$ ... (1)

(iii) If we take $y = a$ as the equation of the directrix and the point S' $(o, -a)$ as the focus the, equation of the parabola $x^2 = -4ay$ ... (2)

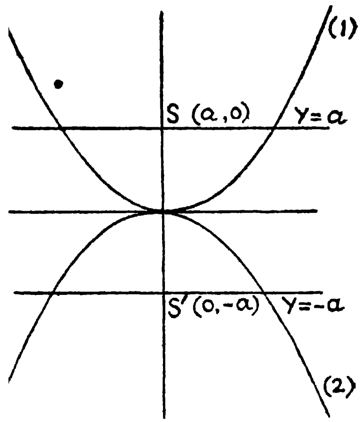


Fig. 28

V-4. Find the equation to the parabola, referred to its axis and the perpendicular line to the axis through the focus, as axes of reference (*i.e.*, x-axis and y-axis).

Here the origin is the focus and A is the vertex.

Take $AS=a$ and $x=-2a$ as directrix.

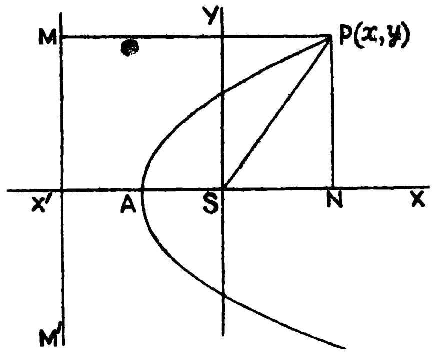


Fig. 29

Let $P\ (x,\ y)$ on the locus, draw PM perpendicular to the directrix MM'. Draw PN perpendicular from P upon OX.

Then $SP=PM=X'N=X'S+SN=2a+x$.

Then $OP^2=x^2+y^2=(x+2a)^2$

$$y^2=(x+2a)^2-x^2=x^2+4ax+4a^2-x^2$$

$\therefore\quad y^2=4a(x+a)$ is the required equation.

V-5. To find the equation to the parabola having the point $(\alpha,\ \beta)$ as focus and the line $lx+my+n=0$ as directrix.

Let $P\ (x,\ y)$ be the moving point. Here $S\ (\alpha,\ \beta)$.

Let $PM=$length of perpendicular from P upon the directrix.

Then $PM=\dfrac{lx+my+n}{\sqrt{l^2+m^2}}$

Now $SP^2=PM^2$

$$\therefore\quad (x-\alpha)^2+(y-\beta)^2=\left\{\frac{lx+my+n}{\sqrt{l^2+m^2}}\right\}^2$$

i.e., $[(x-\alpha)^2+(y-\beta)^2]\ (l^2+m^2)$

$=(lx+my+n)^2$ is the required equation.

V-6. To trace the parabolas

$$(1)\quad x=Ay^2+By+C$$
$$(2)\quad y=Ax^2+Bx+C$$

$$(1)\quad y^2+\frac{B}{A}\ y=x-\frac{C}{A}$$

$$y^2+\frac{B}{A}\ y+\frac{B^2}{4a^2}=\frac{x}{A}-\frac{C}{A}+\frac{B^2}{4A^2}$$

$$\left(y+\frac{B}{2A}\right)^2=\frac{1}{A}\left[x+\frac{B^2-4AC}{4A^2}\right]$$

Change the origin to $\left[-\frac{B^2-4AC}{4A^2},\ -\frac{B}{2A}\right]$

Then $x=x'-\frac{B^2-4AC}{4A^2}$, $y=y'-\frac{B}{2A}$

The equation of the parabola becomes by making x', y' current co-ordinates $y^2=\frac{1}{A}\cdot x$ so that the parabola (1) has vertex $\left(-\frac{B^2-4AC}{4A^2},\ \frac{-B}{2A}\right)$ and latus rectum $=\frac{1}{A}$ and the axis of the parabola is parallel to x-axis.

$$(2)\quad x^2+\frac{B}{A}\ x=\frac{Y}{A}-\frac{C}{A}$$

$$\left(x+\frac{B}{2A}\right)^2=\frac{1}{A}\left(y+\frac{B^2-4AC}{4A^2}\right)$$

changing the origin to $\left(-\frac{B}{2A},\ -\frac{B^2-4AC}{4A^2}\right)$ we get the transformed equation $x^2=\frac{1}{a}\,y$.

Hence the equation represents a parabola whose vertex $\left(-\frac{B}{2A},\ -\frac{B^2-4AC}{4A^2}\right)$ and axis parallel to axis of y

ILLUSTRATIVE EXAMPLES

Ex. Find the equation to the parabola whose vertex is the point $(-1, 1)$ and whose directrix is the straight line $x+y+1=0$

Let P (x, y) be a point on the parabola
$SP^2 = (\text{focal distance})^2 = (x+1)^2+(y-1)^2$.

$(PM)^2 = [\text{distance of the point } (x, y) \text{ from the directrix}]^2$

$$= \left[\frac{x+y+1}{\sqrt{1^2+1^2}}\right]^2 = \left[\frac{x+y+1}{\sqrt{2}}\right]^2$$

$\therefore$ $SP^2 = PM^2$ gives

$$(x+1)^2+(y-1)^2 = \left[\frac{x+y+1}{\sqrt{2}}\right]^2$$

$$x^2+y^2+2x+2y+2 = \tfrac{1}{2}[x^2+y^2+2xy+2y+2x]$$

$$2(x^2+y^2)+4x+4y+4 = x^2+y^2+2xy+2y+2x$$

$$x^2+y^2-2xy+2x+2y+4=0.$$

V-7. The points of intersection of a straight line with the parabola and the condition of tangency.

The co-ordinates of the points of intersection of the straight line $y=mx+c$ and $y^2=4ax$ can be found by solving the equation.

$$\therefore \quad (mx+c)^2 = 4ax.$$

$mx^2+2x(mc-2a)+c^2=0$ gives two values of x corresponding to 2 points of intersection of the line $y=mx+c$ with the parabola.

If the line is tangent, the two points of intersection coincide and hence the equation (1) must have equal roots.

The condition for equal roots

$$\cancel{4}(mc-2a)^2 = \cancel{4}\, m^2 c^2$$

$$m^2c^2-4\,amc+4a^2 = m^2 c^2$$

$$c = \frac{a}{m}$$

Hence $y=mx+\dfrac{a}{m}$ always touches the parabola for all values of m.

Ex. Obtain the equation to tangent to the parabola $y^2=4ax$ which make angle of 60° with x-axis.

Let the equation of tangent $y=mx+\frac{a}{m}$

Now $m=\tan 60=\sqrt{3}$

$\therefore$ The equation of tangent $y=\sqrt{3}.\, x+\frac{a}{\sqrt{3}}$

V-8. To find the equation to the tangent to the parabola $y^2=4ax$ at $(x_1 y_1)$.

Let $P(x_1 y_1)$ $Q(x_2 y_2)$ be any two points on the parabola. The equation to the straight line PQ is

$$y-y_1=\frac{y_2-y_1}{x_2-x_1}(x-x_1) \qquad \cdots \qquad \cdots \qquad \text{(A)}$$

Since $(x_1 y_1)$ $(x_2 y_2)$ are points on the parabola

$$y_2^2=4ax_2$$
$$y_1^2=4ax_1$$
$$\overline{y_2^2-y_1^2=4a(x_2-x_1)} \qquad \therefore \quad \frac{y_2-y_1}{x_2-x_1}=\frac{4a}{y_2+y_1}$$

$\therefore$ from (A), we have

$$y-y_1=\frac{4a}{y_2+y_1}(x-x_1) \qquad \cdots \qquad \cdots \qquad (1)$$

If $(x_1 y_1)$, $(x_2 y_2)$ coincide then PQ becomes tangent, then equation (1) becomes (on putting $y_1=y_2$)

$$y-y_1=\frac{4a}{2y_1}(x-x_1)$$
$$yy_1-y_1^2=2a(x-x_1)$$
$$yy_1=2ax-2ax_1+y_1^2$$
$$=2ax-2ax_1+4ax_1 \qquad [\because \ y_1^2=4ax_1]$$
$$yy_1=2a(x+x_1)$$

Cor. Find the point of the contact of the tangent

$$y=mx+\frac{a}{m}$$

Compairing it with the equation $yy_1=2ax+2ax_1$

$$\frac{y_1}{1}=\frac{2a}{m}=\frac{2ax_1}{\frac{a}{m}} \qquad \therefore \left\{ \begin{array}{l} x_1=\frac{a}{m^2} \\ y_1=\frac{2a}{m} \end{array} \right.$$

$\therefore$ the point of contact is $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$.

Ex. Find the point of the parabola $y^2=12x$ at which the tangent makes 60° with x-axis.

Here $m=\tan 60^\circ=\sqrt{3}$ and $4a=12$

$\therefore$ The point of contact $\left[\frac{3}{(\sqrt{3})^2}, \frac{2.3}{\sqrt{3}}\right]$

or, $[1, 2\sqrt{3}]$.

V-9. To find the equation of the normal of the parabola $y^2=4ax$ at (x_1y_1).

We know the equation to the tangent to the parabola, $y^2=4ax$.

$$yy_1=2a(x+x_1) \quad \cdots \quad \cdots \quad (1)$$

$\therefore$ m of the tangent $=\frac{2a}{y_1}$

Let the equation of the normal at (x_1, y_1)

$$y-y_1=m'(x-x_1) \quad \cdots \quad \cdots \quad (2)$$

$\therefore$ Condition of perpendicularity of (1) and (2)

$$m'\left(\frac{2a}{y_1}\right)=-1 \qquad \therefore \; m'=-\frac{y_1}{2a}$$

$\therefore$ The equation of the normal at (x_1y_1)

$$y-y_1=-\frac{y_1}{2a}(x-x_1)$$

If we take $m=-\frac{y}{2a}$ or, $y_1=-2am$,

then, $4ax_1=y_1^2$ or, $x_1=\frac{y_1^2}{4a}=\frac{4a^2m^2}{4a}=am^2$.

The equation of the normal at $(am^2, -2am)$ becomes $y+2am=m(x-am^2)$ or, $y=mx-2am-am^3$.

Exercises 5

1. Find the equation to the parabola whose focus (2, 3) and whose directrix is the straight line $x-4y+3=0$.

2. Find the equation of the parabola with focus (a, b) and directrix $\frac{x}{a}+\frac{y}{b}=1$.

3. Find vertex, axis and latus rectum and focus to the parabolas (i) $y^2=4x+4y$
(ii) $y^2=4y-4x$.

4. Write down the equation to tangent and normal to the parabola $y^2=9x$ at (4, 6).

5. Write down the equation to tangent and normal to parabola $y^2=12x$ at the end of latus rectum.

6. A tangent to the parabola $y^2=4x$ makes 60° with the axis, find the point of contact.

7. Prove that the straight lines $lx+my+n=0$ touches the parabola $y^2=4ax$ if $ln=am^2$.

8. Find the points of the parabola $y^2=4ax$ at which (i) the tangent and (ii) the normal is inclined at angle 30° to the axis.

9. The parabola $y^2=4px$ goes through the point $(3, -2)$. Obtain the length of latus rectum and the co-ordinates of the focus.

10. Find the equation of the parabola whose focus is at origin and directrix is the straight line $2x+y-1=0$.

11. Prove that the straight line $x+y=1$ touches the parabola $y^2-y+x=0$. Find the point of contact.

12. Find the point of the parabola $y^2=4ax$ at which normal is inclined at 30° with the axis.

13. Prove that $y=\frac{1}{2}x+2a$ is tangent to the parabola $y^2=4ax$.

14. Find the equation to parabola $3y^2=4x$ which is perpendicular to the line $3x+4y+5=0$ and determine the point of contact.

15. (i) Obtain the equation to tangent and normal to the parabola $y^2=4ax$ at the point $(am^2, 2am)$,

(ii) the normal to the parabola at $(am_1^2, 2am_1)$ meet the curve again at $(am_2^2, 2am_2)$. Prove that $m_1^2+m_1m_2+2=0$.

CHAPTER VI

CENTRAL CONIC SECTIONS

Ellipse and Hyperbola

VI-1. An ellipse is a conic section of which the eccentricity e is less than unity and we define it as follows :

An ellipse is the locus of a point which moves in such a manner that its distance from a fixed point, called focus bears a constant ratio e which is less than unity to its perpendicular distance from a fixed line, called the directrix.

A hyperbola is a conic section of which the eccentricity e is greater than unity. We define it as follows :

A hyperbola is the locus of a point which moves in such a manner that its distance from a fixed point, called the focus bears a constant ratio e which is greater than unity to its perpendicular distance from a fixed line, called the directrix.

Both ellipse and hyperbola are central conic sections because any chord (straight line joining any two points on the curve) is bisected at the centre.

VI-2. The equation of an ellipse referred to the directrix and the perpendicular from the focus upon the directrix as axes.

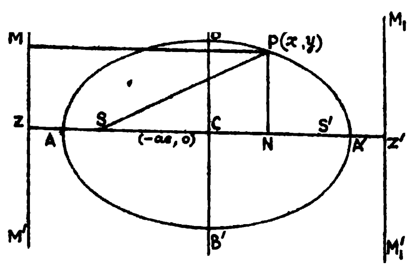


Fig. 80

Let S be the focus, MM' the directrix and e (< 1) the eccentricity of the ellipse.

From S draw SZ perpendicular upon MM'

and let $ZS = c$.

Let ZS and ZM be the axes of reference, so S is the point (c, o).

Take P any point (x, y) on the ellipse and draw PM and PN perpendiculars to ZM and ZS, (produced, if necessary).

Then, from definition,

$$SP = e\,PM = e\,ZN,$$

$$\text{or,}\quad SP^2 = e^2 ZN^2$$

$$\text{or,}\quad (x-c)^2 + y^2 = e^2x^2 \qquad \cdots \qquad (1)$$

which is the required equation.

VI-3. The equation of an ellipse in standard form.

From (1), when $y=0$, $(x-c)^2 = e^2x^2$

$$\text{or,}\quad x - c = \pm ex$$

$$\text{or,}\quad x = \frac{c}{1+e} \quad \text{or,} \quad \frac{c}{1-e}$$

showing that the x-axis meets the ellipse in two points A and A', where $ZA = \dfrac{c}{1+e}$, $ZA' = \dfrac{c}{1-e}$.

Let $AA' = 2a$. $\therefore$ $2a = \dfrac{c}{1-e} - \dfrac{c}{1+e} = \dfrac{2ec}{1-e^2}$.

$$\text{or,}\quad c = (1-e^2)\frac{a}{e} = \frac{a}{e} - ae.$$

If C be the middle point of AA', we have

$$ZC = \tfrac{1}{2}\left(\frac{c}{1+e} + \frac{c}{1-e}\right) = \frac{c}{1-e^2} = \frac{a}{e}.$$

Changing the origin to $C\left(\dfrac{a}{e},\ 0\right)$, the equation (1)

becomes, $\left(x + \dfrac{a}{e} - \dfrac{a}{e} + ae\right)^2 + y^2 = e^2\left(x + \dfrac{a}{e}\right)^2$

$$\text{or,}\quad x^2(1-e^2) + y^2 = a^2(1-e^2).$$

$$\text{or,}\quad \frac{x^2}{a^2} + \frac{y^2}{a^2(1-e^2)} = 1.$$

Now let $a^2(1-e^2)=b^2$ ($\because$ $e<1$)

$\therefore$ The equation becomes

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

Cor. I. $e^2=1-\dfrac{b^2}{a^2}$

Cor. II. The length of the focal chord perpendicular to the axis is called the latus rectum. Its magnitude is obtained thus: If we put $x=ae=cs$ in the equation of the ellipse we get

$$\frac{a^2e^2}{a^2}+\frac{y^2}{b^2}=1 \qquad \frac{y^2}{b^2}=1-e^2=\frac{b^2}{a^2}$$

$$y^2=\frac{b^4}{a^2}$$

So that semilatus rectum $=y=\dfrac{b^2}{a}$

$\therefore$ Latus rectum $=2\dfrac{b^2}{a}$

Cor. III. From the equation to the curve, it is se if (x, y) be a point on the ellipse, the points $(x, -y)$, (x, y) and $(-x, -y)$ are also points on it. Hence the ellipse is symmetrical about both the axes. So the ellipse has a second focus and a second derectrix.

VI-4. The equation of the ellipse having centre (α, β) and axes parallel to the axes of co-ordinates.

Let C (α, β) be the centre of the ellipse, and let the major axis of length $2a$ be parallel to x-axis and the minor axis of length $2b$ be parallel to y-axis. Let (x, y) be co-ordinates of the moving point P with reference to axes parallel to axes of co-ordinates and origin C (α, β)

Then $x=\alpha+X$, $y=\beta+Y$

Now the equation to the ellipse with reference to origin (α, β)

$$\frac{X^2}{a^2}+\frac{Y^2}{b^2}=1$$

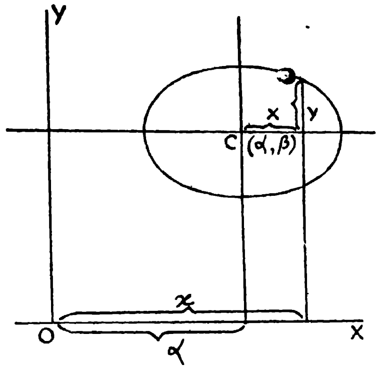


Fig. 81

$\therefore$ The equation of the ellipse with reference to O as origin $\frac{(x-\alpha)^2}{a^2}+\frac{(y-\beta)^2}{b^2}=1.$

VI-5. The general equation of, ellipse.

Let S (α, β) be the focus and $lx+my+n=0$ be the equation of the directrix. Let P (X, Y) be a point on ellipse. Then by definition of ellipse

$SP=ePM$ where PM=Length of perpendicular from the point P upon the straight line $lx+my+nz=0$

$$=\frac{lx+my+n}{\sqrt{l^2+m^2}}$$

$$\therefore \quad SP^2=e^2\,PM^2$$

$$(X-\alpha)^2+(Y-\beta)^2=e^2\left[\frac{lx+my+n}{\sqrt{l^2+m^2}}\right]^2$$

Making (X, Y) current variable co-ordinates we get the equation of the ellipse

$$(l^2+m^2)\,[(x-\alpha)^2+(y-\beta)^2]=e^2\,(lx+my+n)^2$$

ILLUSTRATIVE EXAMPLES

Ex. 1. Find the equation to the ellipse referred to its axes as axes of co-ordinates whose major axis is $\frac{9}{2}$ and eccentricity is $\frac{1}{\sqrt{3}}$. Here $a=\frac{9}{4}$

We know $e^2=1-\frac{b^2}{a^2}$

$$\left(\frac{1}{\sqrt{3}}\right)^2=1-\frac{b^2}{\frac{9^2}{4^2}}=1-\frac{16b^2}{81}$$

$$\text{or,}\quad \frac{16b^2}{81}=1-\frac{1}{3} \qquad \therefore\quad b^2=\frac{2}{\cancel{3}}\times \overset{27}{\cancel{81}}\times\frac{1}{16}=\frac{27}{8}$$

$\therefore$ Equation of ellipse $\frac{x^2}{\frac{81}{16}}+\frac{y^2}{\frac{27}{8}}=1$

$\frac{16x^2}{81}+\frac{8y^2}{27}=1$ $\therefore$ $16x^2+24y^2=81$ is eqn. of ellipse.

Ex. 2. Find the equation to the ellipse whose focus is $(-1, 1)$ and directrix is $x-y+3=0$ and eccentricity is $\frac{1}{2}$.

Let $P\ (x, y)$ be the moving point

Then $SP=ePM$

$SP^2=e^2PM^2$ gives

$$(x+1)^2+(y-1)^2=(\tfrac{1}{2})^2\left[\frac{x-y+3}{\sqrt{1^2+1^2}}\right]^2$$

$$=\frac{1}{4}\cdot\frac{(x-y+3)^2}{2}$$

$$8[x^2+y^2+2x-2y+2]=(x-y+3)^2$$

$$=x^2+y^2-2xy+6x-6y+9$$

$$7x^2+7y^2+2xy+10x-10y+7=0$$

Ex. 3. Find the latus rectum, eccentricity and co-ordinates of the foci of the following ellipses :

(i) $16x^2+9y^2=144$ (ii) $4x^2+3y^2-6y-9=0$

(i) The equation can be reduced to the form

$$\frac{x^2}{9}+\frac{y^2}{16}=1$$

So that $a^2=9, \quad b^2=16$ [semi major axis$=4$
semi minor axis$=3$]

$$e^2=1-\frac{a^2}{b^2}=1-\frac{9}{16}=\frac{7}{16}$$

$$e=\frac{\sqrt{7}}{4}$$

Latus rectum$=2\frac{a^2}{b}=2\cdot\frac{9}{4}=\frac{9}{2}$

Here major axis is $x=0$

Co-ordinates of foci $(0, \pm eb)$

$$\left(0, \pm\frac{\sqrt{7}}{4}\times 4\right) \quad \text{or,} \quad (0, \pm\sqrt{7})$$

(ii) $4x^2+3y^2-6y=9$

$4x^2+3(y^2-2y+1)=12$

$\frac{x^2}{3}+\frac{(y-1)^2}{4}=1$ so that centre (0, 1)

$e^2=1-\frac{3}{4}=\frac{1}{4} \qquad e=\frac{1}{2}$

Changing the origin to (0, 1), the equation of ellipse

$$\frac{X^2}{3}+\frac{Y^2}{4}=1$$

Foci $(0, \frac{1}{2}\cdot 2)$

or, (0, 1) referred to the new origin

$\therefore$ Foci (0, 0) referred to the old origin

$\therefore$ The focus is at the origin.

VI-6. The standard equation of the hyperbola.

Let S be the focus and MZ the directrix, e the eccentricity.

Take $CA=a$ since $e>1$,

and $CZ=\frac{a}{e} \qquad CS=ae$

Z will be on the left side of A, and S will be on the right side of A and $\frac{SA}{AZ}=\frac{SA'}{A'Z}=e$

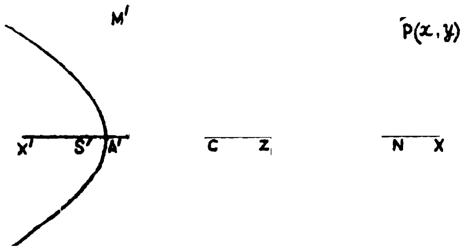


Fig. 82

By definition of hyperbola $SP=ePM$

$$SP^2=e^2PM^2 \qquad \text{and } S\ (ae,\ 0)$$

$$(x-ae)^2+y^2=e^2\ ZN^2=\text{if } P \text{ be the point } (x,\ y)$$

$$=e^2(CN-CZ)^2$$

$$=e^2\left(x-\frac{a}{e}\right)^2$$

$$x^2-2aex+a^2e^2+y^2=e^2x^2-2aex+a^2$$

$$a^2(e^2-1)=x^2(1-e^2)-y^2$$

$$1=\frac{x^2}{a^2}-\frac{y^2}{a^2(e^2-1)}$$

Taking $b^2=a^2\ (e^2-1)$

The equation becomes $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

and $\frac{b^2}{a^2}=e^2-1$ which is positive $\because\ e>1.$

So that $e^2=1+\frac{b^2}{a^2}$

From the equation $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

$$a^2 \quad a^2(e^2-1)$$

$$x^2(e^2-1)-y^2=a^2(e^2-1)$$

or, $x^2+a^2e^2+y^2=a^2+e^2x^2$

or, $(x+ae)^2+y^2=e^2\left(x+\frac{a}{e}\right)^2$

i.e., $(CN+CS')^2+PN^2=e^2(CN+CZ')^2$

$$S'N^2+PN^2=e^2Z'N^2$$

$$S'P^2=e^2PM'^2$$

Distance of P from $S'=ex$ distance of P from $Z'M'$. Hence the curve might have described with S' as focus and $Z'M'$ as directrix.

So that the hyperbola has a second focus and a second directrix.

The line AA' is called the transverse axis of the hyperbola. If two points B and B' are taken on the y-axis such that $CB=CB'=b$, then the length BB' is conjugate axis of the hyperbola.

Latus rectum is the length of the focal chord perpendi cular to the transverse axis.

From $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

when $x=ae-\frac{y^2}{b^2}=1-\frac{x^2}{a^2}=1-\frac{a^2e^2}{a^2}$

$$-y^2=(1-e^2)b^2$$

$$y^2=b^2(e^2-1)=b^2\frac{b^2}{a^2}$$

Semi latus rectum $=y=\frac{b^2}{a}$

$\therefore$ Latus rectum $=\frac{2b^2}{a}$

VI-7. (*a*) The condition of tangency of line $y=mx+c$ to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

The co-ordinates of the points of intersection of line $y=mx+c$ and ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ can be obtained by solving.

$\therefore$ The x-co-ordinates of the points of intersection are roots of $\frac{x^2}{a^2}+\frac{(mx+c)^2}{b^2}=1$

$$b^2x^2+a^2(m^2x^2+2mxc+c^2)=a^2b^2$$

$$x^2(a^2m^2+b^2)+2mxa^2c+a^2(c^2-b^2)=0 \quad \cdots \quad (1)$$

The line will touch the ellipse, if two roots of (1) are equal.

Then $(2mca^2)^2-4a^2(c^2-b^2)(a^2m^2+b^2)=0$

or, $m^2c^2a^4-a^2(a^2m^2c^2+b^2c^2-a^2b^2m^2-b^4)=0$

or, $m^2c^2a^2-m^2c^2a^2-b^2c^2+a^2b^2m^2+b^4=0$

$$b^2(a^2m^2+b^2)=b^2c^2$$

$c^2=a^2m^2+b^2$ which is the condition of tangency.

Putting value of c, in the equation $y=mx+c$ we have $y=mx+\sqrt{a^2m^2+b^2}$ always touches the hyperbola.

(b) The condition of tangency of the line $y=mx+c$ to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ can be found by changing $-b^2$ for b^2 in the above article.

$\therefore$ the condition of tangency $c^2=a^2m^2-b^2$.

VI-8. Find the equation of tangents to (i) the ellipse (ii) the hyperbola at (x_1y_1).

(i) Let $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ be equation of ellipse.

Let (x_1y_1) (x_2y_2) be the any equation of the chord of the ellipse.

Since (x_1y_1) is a point on ellipse $\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}=1$

,, (x_2y_2) ,, ,, $\frac{x_2^2}{a^2}+\frac{y_2^2}{b^2}=1$

$$\therefore \quad \frac{(x_2^2-x_1^2)}{a^2}=-\frac{y_2^2-y_1^2}{b^2}$$

$$\frac{(y_2-y_1)(y_2+y_1)}{b^2}=-\frac{(x_2-x_1)(x_2+x_1)}{a^2}$$

$$\therefore \quad \frac{y_2-y_1}{x_2-x_1}=-\frac{b^2}{a^2}\cdot\frac{x_2+x_1}{y_2+y_1}$$

Now, equation of PQ, a line joining (x_1y_1) and $x_2y_2)$

$$\text{is} \quad y-y_1=\frac{y_2-y_1}{x_2-x_1}(x-x_1) \quad \cdots \quad (1)$$

$$\therefore \quad y-y_1=-\frac{b^2}{a^2}\cdot\frac{x_2+x_1}{y_2+y_1}(x-x_1)$$

If PQ be tangent, two points P, Q coincide ultimately, o that $(x_2=x_1,\ y_2-y_1)$. So that (1) becomes

$$y-y_1=-\frac{b^2}{a^2}\cdot\frac{2x_1}{2y_1}(x-x_1)$$

$$\frac{yy_1}{b^2}-\frac{y_1^2}{b^2}=-\frac{xx_1}{a^2}+\frac{x_1^2}{a^2}$$

$$\therefore \quad \frac{xx_1}{a^2}+\frac{yy_1}{b^2}=\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}=1$$

[$\because$ (x_1y_1) is a point on ellipse]

$\therefore$ Equation of tangent at (x_1y_1)

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1 \quad \cdots \quad \cdots \quad (2)$$

(ii) Similarly we can find the equation of tangent to hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

at (x_1y_1) by putting $-b^2$ for b^2 in (2), $\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1$

VI-9. Auxiliary circle and eccentric angle.

If we draw a circle with major axis AA' as diameter, then the circle is called the auxiliary circle. If the ordinate of any point P *viz.* PN be produced to meet the auxiliary circle in Q, then $\angle QOA=\theta=$eccentric angle of P.

$$ON=a\cos\theta=x\text{-co-ordinate of } P$$

If we put $x=a\cos\theta$ in $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

$$\frac{x^2\cos^2\theta}{a^2}+\frac{y^2}{b^2}=1$$

$$\frac{y^2}{b^2}=1-\cos^2\theta=\sin^2\theta$$

$$y=b\sin\theta.$$

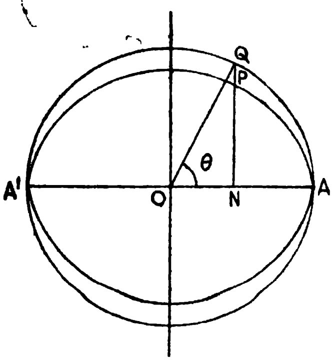


Fig. 88

Thus if θ be the eccentric angle of the point P, then co-ordinates of P are $(a\cos\theta, b\sin\theta)$.

Thus the equation of tangent, in terms of θ, the eccentric angle can be obtained by putting $x_1=a\cos\theta$, $y_1=b\sin\theta$ in the equation of tangent $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$

$$\frac{x\,.\,a\cos\theta}{a^2}+\frac{y\,.\,b\sin\theta}{b^2}=1$$

$$\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1.$$

VI-10. To find the equation to the normal (i) to ellipse (ii) to the hyperbola.

We have the equation of tangent at x_1y_1

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$$

$$\frac{yy_1}{b^2}=-\frac{xx_1}{a^2}+1 \qquad (1)$$

so that m of the tangent $=-\frac{x_1}{y_1}\cdot\frac{b^2}{a^2}$

Let the equation of normal at (x_1y_1), a straight line through the point (x_1y_1)

$$y-y_1=m'(x-x_1) \quad \cdots \quad \cdots \quad (2)$$

$\therefore$ Condition that (1) & (2) are perpendicular

$$m'\left(-\frac{x_1}{y_1}\cdot\frac{b^2}{a^2}\right)=-1$$

$$m'=\frac{a^2}{b^2}\cdot\frac{y_1}{x_1}$$

$\therefore$ Equation of normal at $(x_1 y_1)$

putting $m'=\frac{a^2}{b^2}\cdot\frac{y_1}{x_1}$ in (2) becomes,

$$y-y_1=\frac{a^2}{b^2}\cdot\frac{y_1}{x_1}(x-x_1)$$

$$\frac{y-y_1}{\frac{y_1}{b^2}}=\frac{x-x_1}{\frac{x_1}{a^2}}$$

or, $\frac{a^2x}{x_1}-\frac{b^2y}{y_1}=a^2-b^2$

(ii) Changing b^2 to $-b^2$, the equation of normal at (x_1, y_1) on the hyperbola is $\frac{a^2x}{x_1}+\frac{b^2y}{y_1}=a^2+b^2$.

[If we take $(x_1\ y_1)$ as $(a\cos\theta, b\sin\theta)$, so that θ is the eccentric angle of the point.

The equation of normal at $(a\cos\theta, b\sin\theta)$ on the ellipse becomes

$$-\frac{y-b\sin\theta}{\frac{b\sin\theta}{b^2}}=\frac{x-a\cos\theta}{\frac{a\cos\theta}{a^2}}$$

by cosec $\theta-b^2=ax\sec\theta-a^2$

$\therefore$ Equation of normal

$ax\sec\theta-by\operatorname{cosec}\theta=a^2-b^2$]

ILLUSTRATIVE EXAMPLES

Ex. 1. Find the points on ellipse $2x^2+3y^2=11$ at which tangent is parallel to $4x-3y+5=0$

The equation of ellipse, $\frac{x^2}{\frac{11}{2}}=\frac{y^2}{\frac{11}{3}}=1$

So that $a^2=\frac{11}{2}$, $b^2=\frac{11}{3}$

Equation of any tangent.

$$y=mx+\sqrt{a^2 m^2+b^2}$$

Here $y=mx+\sqrt{\frac{11}{2} m^2+\frac{11}{3}}$

This is parallel to

$$3y=4x+5 \quad \text{or,} \quad y=\tfrac{4}{3}x+\tfrac{5}{3}.$$

So that $m=\frac{4}{3}$

∴ The equation of tangent

$$y=\tfrac{4}{3}x+\sqrt{\tfrac{11}{2}\times\tfrac{16}{9}+\tfrac{11}{3}}$$

$$y=\frac{4x}{3}\times\frac{11}{3}$$

$$4x-3y=11$$

$$\frac{4x}{11}-\frac{3y}{11}=1 \quad \cdots \quad \cdots \quad \cdots \quad (1)$$

The tangent at $x_1\ y_1$ to ellipse

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1 \quad \text{putting } a^2=\frac{11}{2},\ b^2=\frac{11}{3}$$

or, $\dfrac{2xx_1}{11}+\dfrac{3yy_1}{11}=1 \quad \cdots \quad \cdots \quad \cdots \quad (2)$

Compairing (1) & (2) $\dfrac{4}{11}=\dfrac{2x_1}{11}$, ∴ $x_1=2$

$$-\frac{3}{11}=\frac{3y_1}{11} \quad \therefore \quad y_1=-1$$

∴ Point of contact $(2, -1)$

Ex. 2. Find the equation of tangent and normal to ellipse $3x^2+4y^2=48$ at $(2, 3)$.

Equation of ellipse $\dfrac{x^2}{16}+\dfrac{y^2}{12}=1$ ∴ $\begin{array}{l} a^2=16 \\ b^2=12 \end{array}$

Now equation of tangent at $(x_1\ y_1)$

$$\frac{xx_1}{16}+\frac{yy_1}{12}=1$$

The equation of tangent at (2, 3)

$$\frac{2x}{16}+\frac{3y}{12}=1$$

or, $x+2y=8$ is the required equation of tangent.

The equation of normal at $(x_1\ y_1)$

$$\frac{y-y_1}{\frac{y_1}{b^2}}=\frac{x-x_1}{\frac{x_1}{a^2}}$$

$$\frac{y-y_1}{\frac{y_1}{12}}=\frac{x-x_1}{\frac{x_1}{16}}$$

$\therefore$ Equation of normal at (2, 3)

$$\frac{y-3}{\frac{3}{12}}=\frac{x-2}{\frac{2}{16}}$$

or, $4y-12=8x-16$

Divide by 4, $2x-y=1$ is the required equation of normal.

Ex. 3. Find the equation of the ellipse (referred to its axes as the axes of x, y respectively) which passes through the point $(-3, 1)$ and has eccentricity $\sqrt{\frac{2}{5}}$.

Let the equation of ellipse be

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

Since it passes through $(-3, 1)$ we have

$$\frac{9}{a^2}+\frac{1}{b^2}=1 \quad \cdots \quad \cdots \quad (1)$$

Also we have the relation $e^2=1-\frac{b^2}{a^2}$ and $e=\sqrt{\frac{2}{5}}$

$$\frac{2}{5}=1-\frac{b^2}{a^2} \quad \therefore \quad \frac{b^2}{a^2}=1-\frac{2}{5}=\frac{3}{5}$$

or, $b^2=\frac{3}{5}a^2$

or, $\frac{1}{b^2}=\frac{5}{3}\cdot\frac{1}{a^2}$

$\therefore$ From (1) $\frac{9}{a^2}+\frac{5}{3}\cdot\frac{1}{a^2}=1$

$$a^2=\frac{32}{3} \text{ and } b^2=\frac{32}{5}$$

$\therefore$ The required equation of the ellipse

$$\frac{x^2}{\frac{32}{3}}+\frac{y^2}{\frac{32}{5}}=1$$

or, $3x^2+5y^2=32$

Ex. 4. Find the equation of the hyperbola if distance between 2 foci $=16$ and eccentricity $=\sqrt{2}$.

Now, $2ae=$ distance between 2 foci $=16$

$$ae=8$$

$$\therefore \quad a=\frac{8}{e}=\frac{8}{\sqrt{2}}=4\sqrt{2}.$$

Again, $\frac{b^2}{a^2}=e^2-1=2-1=1$

$$\therefore \quad b^2=a^2=32$$

$\therefore$ The equation is, $\frac{x^2}{32}-\frac{y^2}{32}=1$

or, $x^2-y^2=32$.

Ex. 5. Find the equations of tangents to the ellipse $x^2+9y^2=2$ which is perpendicular to the line $3x+4y+1=0$.

Equation of the ellipse $\frac{x^2}{2}+\frac{y^2}{\frac{2}{9}}=1$

The equation of axes tangent $y=mx\pm\sqrt{a^2m^2+b^2}$,

taking $a^2=2$, $b^2=\frac{2}{9}$ we get, $y=mx\pm\sqrt{2m^2+\frac{2}{9}}$ ...(1)

This is perpendicular to $4y=-3x+1$

or, $y=-\frac{3}{4}x+\frac{1}{4}$ ... ...(2)

$\therefore$ The condition of tangency of lines (1) and (2)

$$m\left(-\tfrac{3}{4}\right)=-1$$

$$\therefore \quad m=\tfrac{4}{3}$$

Hence the equation of tangent (1) becomes on putting $m=\frac{4}{3}$

$$y=\tfrac{4}{3}x\pm\sqrt{2\times\tfrac{16}{9}+\tfrac{2}{9}}$$

$$y=\frac{4}{3}x\pm\frac{\sqrt{34}}{3}$$

$3y=4x\pm\sqrt{34}$ is the required equation of the tangent.

Exercises 6

1. Find the co-ordinates of foci, eccentricity and equation of directrixes of the following ellipses :

(a) $9x^2+25y^2=225$

(b) $4x^2+5y^2=1$

2. Find the equation of the ellipse whose latus rectum is 5 and eccentricity is $\frac{2}{3}$. [C. U.]

3. Find the equation to the ellipse whose centre is at $(-2, 3)$ and whose axes are 3 and 2 and whose major axis is parallel to the axis of x.

4. Find the eccentricity of an ellipse, if its latus rectum be equal to $\frac{1}{2}$ its minor axis.

5. Find the equation of the ellipse whose foci are $(4, 0)$ and $(-4, 0)$ and whose eccentricity $=\frac{1}{3}$.

6. Find the equation of the ellipse whose focus is the point $(-1, 1)$ and whose directrix $x-y+3=0$ and whose eccentricity is $\frac{1}{2}$. [C. U. 1952]

7. Find the equation of tangent and normal at $(1, \frac{4}{3})$ of the ellipse $4x^2+9y^2=20$.

8. Find the equation of tangent and normal to the ellipse $9x^2+16y^2=144$ at end of latus rectum.

9. Prove that the straight line $y=x+\sqrt{\frac{7}{12}}$ touches the ellipse $3x^2+4y^2=1$.

10. Find the equation to the tangent of the ellipse $4x^2+3y^2=5$ which are parallel to the straight line $y=3x+7$.

11. Find the tangent and normal to the ellipse $2x^2+9y^2=12$ at the point $(2, \frac{2}{3})$.

12. Find the equations of tangents to the ellipse $2x^2+3y^2=1$ which are parallel to the line $2x-y+3=0$.

13. Find the eccentric angle of a point on the ellipse $\frac{x^2}{5}+\frac{y^2}{4}=2$ at a distance 3 from the centre.

14. Find the equation to the hyperbola whose conjugate axis is 5 and distance between whose foci is 13.

15. Find the equation to the hyperbola whose conjugate axis is 7 and which passes through $(3, -2)$.

16. In the hyperbola $4x^2 - 9y^2 = 36$, find the axis, the co-ordinates of their foci, the eccentricity and the latus rectum.

17. Find the equations of tangents to the hyperbola which is parallel to line $4y = 5x + 7$.

18. Find the equation of tangents to hyperbola $x^2 - 5y^2 = 40$ which is perpendicular to line $2x - y + 3 = 0$.

19. Calculate the eccentricity and the position of two foci of the hyperbola $\frac{x^2}{12^2} - \frac{y^2}{5^2} = 1$. [C. U. 1957]

20. Find the equation to the hyperbola whose foci are $(-1, 3)$ and $(5, 3)$ and whose eccentricity is $\frac{3}{2}$.

21. Satisfy yourselves that the line $5x - 3y = 8\sqrt{2}$ is a normal to the ellipse $\frac{x^2}{25} + \frac{y^2}{9} = 1$.

22. Find the equations to tangent to ellipse $x^2 + 9y^2 = 2$ which are perpendicular to $3x + y + 1 = 0$. Find the point of contact.

APPENDIX (A)

1. The equation of the chord of an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ joining $(x_1, y_1)(x_2, y_2)$.

Equation of any straight line, joining $(x_1\ y_1)\ (x_2\ y_2)$ is given by

$$\frac{y-y_1}{y_2-y_1}=\frac{x-x_1}{x_2-x_1} \quad \ldots \quad (1)$$

Also since $(x_1\ y_1)\ (x_2\ y_2)$ are point on ellipse

$$\frac{x_2^2}{a^2}+\frac{y_2^2}{b^2}=1$$

$$\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}=1$$

$\therefore$ by subtraction, $\frac{x_2^2-x_1^2}{a^2}+\frac{y_2^2-y_1^2}{b^2}=0$

$$\frac{(y_1+y_2)\ (y_2-y_1)}{}=-\frac{(x_2-x_1)\ (x_2+x_1)}{} \quad (2)$$

Multiplying (1) by (2), we get equation of chord joining $(x_2\ y_2)\ (x_1, y_1)$, points of the ellipse

$$\frac{(y-y_1)\ (y_1+y_2)}{b^2}=-\frac{(x-x_1)\ (x_2+x_1)}{a^2} \quad \ldots \quad (3)$$

2. The equation of chord having (α, β) as its middle point.

If (α, β) be middle point of joins of $(x_1\ y_1)\ (x_2, y_2)$

then $\alpha=\frac{x_1+x_2}{2}$, $\beta=\frac{y_1+y_2}{2}$

From the equation (3)

$$\frac{y-y_1}{b^2}\ 2\beta+\frac{(x-x_1)}{a^2}\ 2\alpha=0$$

or, $$\frac{x\alpha}{a^2}+\frac{y\beta}{b^2}=\frac{x_1\alpha}{a^2}+\frac{y_1\beta}{b^2} \quad \ldots \quad (4)$$

Since (α, β) is a point on this straight line then putting α, β for x, y in (4) $\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}=\frac{x_1\alpha}{a^2}+\frac{y_1\beta}{b^2}$.

So that the equation (4) becomes $\frac{x\alpha}{a^2}+\frac{y\beta}{b^2}=\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}$.

Cor. Similarly the equation of chord of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at $(\alpha, \beta) = 1$ is given by

$$\frac{x\alpha}{a^2}-\frac{y\beta}{b^2}=\frac{\alpha^2}{a^2}-\frac{\beta^2}{b^2}.$$

3. *Diameter and conjugate diameter.*

The locus of middle points of system of chords parallel to the straight line $y=mx$ is a straight line passing through the centre. Such straight line is called the diameter of ellipse with regards to such chords.

If $(\alpha\beta)$ be middle of a chord, then equation to such chord is

$$\frac{x\alpha}{a^2}+\frac{y\beta}{b^2}=\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}.$$

Now this chord is parallel to $y=mx$

$$\text{Therefore, } m=-\frac{\alpha}{\beta}\cdot\frac{b^2}{a^2}$$

Making (α, β) current, the locus of middle point $(\alpha\beta)$ of system of chords parallel to $y=mx$ is

$$m=-\frac{x}{y}\,\frac{b^2}{a^2}$$

$\therefore\quad y=-\frac{b^2}{a^2m}x$ (i) which is the diameter with regards to chords parallel to $y=mx$

If equation of (i) can be put in the form $y=m'x$

So that $\quad m'=-\frac{b^2}{a^2m}\qquad\therefore\quad mm'=-\frac{b^2}{a^2}$

The symmetry of this result proves the following :

$y=m'x$ be the locus of middle points of all chords parallel to $y=mx$, where $mm'=-\frac{b^2}{a^2}$ and $y=mx$ be locus of middle points of all chords parallel to $y=m'x$.

$\therefore$ Thus diameters $y=mx$, $y=m'x$ are such that each bisects all chords parallel to the other, such straight lines $y=mx$, $y=m'x$ are called the conjugate diameters.

4. *The equation of auxiliary circle.*

$x^2+y^2=a^2$ represents a circle with the major as diameter called the auxiliary circle.

5. *The equation of director circle of ellipse.*

The equation of any tangent

$$y = mx + \sqrt{a^2m^2 + b^2} \quad \ldots \quad (1)$$

Equation to any straight line, perpendicular to (1) will be found by taking

$$m' = -\frac{1}{m} \qquad \text{or, } m = -\frac{1}{m'}$$

$\therefore$ The equation of tangent perpendicular to (1)

$$y = -\frac{1}{m}x = \sqrt{\frac{a^2}{m^2} + b^2} \quad \ldots \quad (2)$$

$$\text{or, } my + x = \sqrt{a^2 + b^2m^2} \quad \ldots \quad (3)$$

$$\text{and, from (1), } y - mx = \sqrt{a^2m^2 + b^2} \quad \ldots \quad (4)$$

Squaring and adding (3) and (4)

$$y^2(1+m^2) + x^2(1+m^2) = a^2m^2 + b^2 + a^2 + b^2m^2$$
$$= a^2(1+m^2) + b^2(1+m^2)$$

$\therefore$ Locus of points of intersection of two perpendicular tangents is given by

$x^2 + y^2 \equiv a^2 + b^2$ which is equation of director circle.

Cor. Similarly by the equation of the director circle of a hyperbola can be easily obtained by putting $-b^2$ for b^2.

$\therefore$ The equation of director circle of hyperbola

$$x^2 + y^2 = a^2 - b^2.$$

APPENDIX (B)

1. *Different kinds of hyperbola.*

In the hyperbola, $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, if the transverse axis and conjugate axis are interchanged the hyperbola is called conjugate hyperbola.

Thus $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$ or, $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$

is the equation of the conjugate hyperbola.

Ex. If e and e' be eccentricity of a hyperbola and its conjugate show that $\frac{1}{e^2} + \frac{1}{e'^2} = 1$

Let $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ and $\frac{y^2}{b^2}-\frac{x^2}{a^2}=1$

be equations of a hyperbola and its conjugate

$$e'^2=1+\frac{a^2}{b^2}=\frac{a^2+b^2}{b^2}$$

So that $\frac{1}{e^2}+\frac{1}{e'^2}=\frac{a}{a^2+b^2}+\frac{b^2}{a^2+b^2}=\frac{a^2+b^2}{a^2+b^2}=1.$

2. *Asymptotes.*

Asymptote is defined to be a straight line lying not wholly at infinity cutting the curve in two coincident points at infinity.

Let $y=mx+c$ be equation of the asymptote to hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

Then $\frac{x^2}{a^2}-\frac{(mx+c)^2}{b^2}=1$ gives the x-co-ordinates of the points of intersection of the asymptote with the curve hence must have infinite values of x. Then coefficient of x^2 and x are separately zero.

$$\frac{1}{a^2}-\frac{m^2}{b^2}=0 \quad \text{and} \quad \frac{2mc}{b^2}=0 \quad \therefore \quad m=\pm\frac{b}{a}, \quad c=0$$

Hence $y=\pm\frac{b}{a}x$ are asymptotes, combining two equations

$$\left(\frac{x}{a}+\frac{y}{b}\right)\left(\frac{x}{a}-\frac{y}{b}\right)=0 \quad \text{or,} \quad \frac{x^2}{a^2}-\frac{y^2}{b^2}=0.$$

3. *Rectangular hyperbola.*

If two asymptotes of a hyperbola are perpendicular, then the hyperbola is called rectangular hyperbola.

We have $y=\frac{b}{a}x$, $y=-\frac{b}{a}x$ as equation of two asymptotes.

If $\frac{b}{a}\left(-\frac{b}{a}\right)=-1$ $b^2=a^2$, hence the equation of the rectangular hyperbola is $\frac{x^2}{a^2}-\frac{y^2}{a^2}=1$ or, $x^2-y^2=a^2$

(eccentricity of rectangular hyperbola) is given by

$$e^2=1+\frac{b^2}{a^2}=1+\frac{a^2}{a^2}=2 \quad \therefore \quad e=\sqrt{2}.$$

SOLID GEOMETRY

DEFINITIONS

1. (i) A **point** has *no magnitude*, that is to say, neither length, breadth nor thickness.

(ii) A **line** has *length* without breadth or thickness.

(iii) A **surface** has *length* and *breadth* without thickness.

(iv) A **solid** has *length, breadth* and *thickness.*

Hence, a point is said to be of *no dimension*
a line ... ... of *one dimension*
a surface ... ... of *two dimensions*
a solid ... ... of *three dimensions.*

2. A solid geometry treats of properties of solids, surfaces, lines and points in space and also of their mutual relations. It is also called the geometry of three dimensions.

3. It is seen from the definitions that solids, surfaces and points are related to one another :

(a) Solids are bounded by surfaces.

(b) Surfaces are bounded by lines and they meet in lines.

(c) Lines are bounded by points and meet in points.

4. A **plane** is a surface such that the straight line joining any two points in it lies wholly in the surface.

5. Straight lines are said to be **coplanar** if they lie on the same plane. Two coplanar straight lines may intersect at a point or may not intersect, if produced indefinitely beyond both ends. Two straight lines are said to be *parallel* when they are coplanar and do not meet if produced indefinitely beyond both ends.

6. Straight lines are to be **skew** if no plane can be made to pass through them.

7. **Planes** are said to be parallel when they do not meet though extended indefinitely.

8. A *straight line* and a *plane* are said to be *parallel* when they do not meet though both are indefinitely extended.

9. A straight line is said to be *perpendicular* or *normal to a plane* when it is perpendicular to every straight line which meets it in that plane.

Thus the st. line OP is perp. to the plane XY if it be perp. to the st. lines OA, OB, $OC\cdots$ all drawn through the point O in the XY.

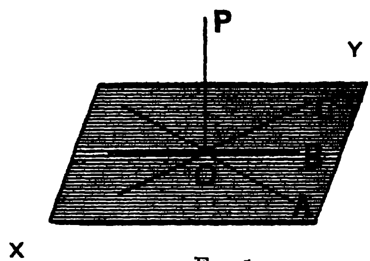


Fig. 1

10. When two planes intersect along a st. line, they are said to form a **dihedral angle.** A dihedral angle is measured by the plane angle contained by two st. lines drawn from *any* point in the line of section at right angles to it, one in each plane.

Thus, the planes AB and CD intersect along the straight line AC. From any point O in AC, st. lines OP, OQ are drawn perp. to AC, one in each plane. The dihedral angle formed by the planes is measured by the $\angle POQ$.

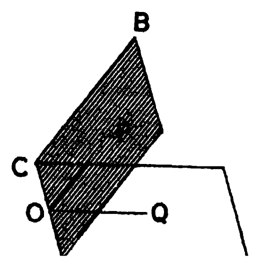


Fig. 2

11. **The projection of a point** on a plane is the foot of the perpendicular from the point to the plane. Here the projection as defined is called the orthogonal projection.

The projection of a line on a plane is the locus of the feet of perpendiculars drawn from all points on the given line to the plane.

It can be proved that the projection of a straight line on a plane is itself a straight line. The angle which a straight line makes with a plane is measured by the angle between the straight line and its projection on the plane. The length of the projection of a straight line AB on a plane is $AB \cos\theta$, where θ is the angle which the line makes with the plane.

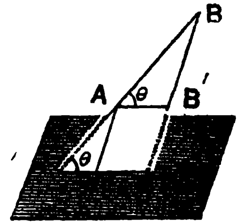


Fig. 8

For, $ab = AB' = AB \cos\theta$.

Axioms

1. *One and only one plane passes through a given line and a given point outside it.*

2 *If two planes have one point in common, they have at least a second point in common.*

From the first axiom, it at once follows that *one and only one plane may be drawn through two intersecting straight lines.*

For, let the two straight lines AB and CD intersect at O.

Now C being a point outside the straight line AB, one and only one plane may be drawn through the straight line AB and the point C. Again the point C and the point O in AB, being in the plane, the straight line CD must lie in the plane so drawn.

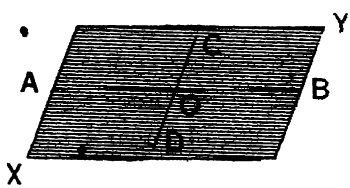


Fig 4

Theorem 1

Two intersecting planes cut one another in a straight and in no point outside it.

Let PQ and XY be two intersecting planes.

It is required to prove that the planes PQ and XY cut one another in a straight line and in no point outside it.

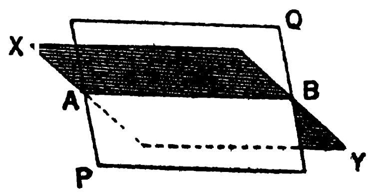


Fig. 5

Proof. Let A and therefore B be two points lying on the common section of the two planes.

Since A and B lie in both planes PQ and XY, the straight line AB lies wholly in both the planes, that is, the planes intersect along a straight line.

There can be no other point outside the straight line AB common to both the planes. If possible, let C be such a point. So two different planes can be drawn through the straight line AB and the point C outside it, which is impossible.

Hence two intersecting planes cut one another in a straight line and in no point outside it. *Q. E. D.*

Exercises 1

1. Show that any number of straight lines through a given point and intersecting a given straight line are coplanar. [C. U. 1954]

2. If any number of parallel lines be cut by a common transversal, they are coplanar. [C. U. 1915, '21]

3. Prove that any three planes, which do not pass through a common straight line, cut one another in straight lines which either meet at a point or are parallel. [C. U. 1911, '24]

Theorem 2

If a straight line is perpendicular to each of two intersecting straight lines at their point of intersection, it is perpendicular to the plane in which they lie.

Let the straight line OP be perpendicular to each of the two intersecting straight lines OA and OB at their common point of intersection O. Let XY be the plane in which OA and OB lie.

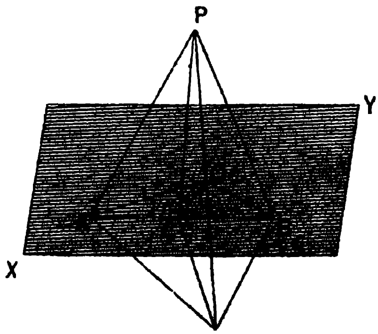


Fig. 6

It is required to prove that OP is perpendicular to the plane XY.

In the plane XY draw *any* line OC through O and draw the straight line AB to cut OC in C. Produce PO to Q, making OQ equal to OP. Join PA, PB, PC and QA, QB, QC by straight lines.

Proof. $\triangle AOP \equiv \triangle AOQ$,

$\because$ $OP = OQ$, OA is common and $\angle AOP = \angle AOQ$, each being a right angle.

$\therefore$ $AP = AQ$.

Similarly, $BP = BQ$.

$\because$ $\triangle APB \equiv \triangle AQB$.

[$\because$ $AP = AQ$, $BP = BQ$, and AB is common]

Hence, the $\angle PAB =$ the $\angle QAB$ *i.e.* $\angle PAC = \angle QAC$.

Now in the $\triangle^s PAC$, QAC,

$\because$ $PA = QA$, AC is common, and $\angle PAC = \angle QAC$

$\therefore$ the $\triangle^s$ are equal in all respects and so $PC = QC$.

Hence in the $\triangle^s POC$, QOC,

Since $PO = QO$, $PC = QC$ and OC is common.

$\therefore$ the $\triangle^s$ are equal in all respects.

$\therefore$ the $\angle POC =$ the $\angle QOC =$ a right angle

$\therefore$ OP is perpendicular to any straight line OC which meets it in the plane XY, that is, OP is perpendicular to the plane of OA and OB. *Q. E. D.*

Exercises 2

1, Find the locus of a point in space equidistant from

(i) two given points. [C. U. 1915, '28, '39, '47]

(ii) three given non-collinear points. [C. U. 1941]

2. Show that all points on the circumference of a circle are equidistant from any point on the normal to the plane of the circle through the centre. [C. U. 1937]

3. OA, OB, OC are three straight lines which are mutually perpendicular. If AD is drawn perpendicular to BC, show that OD is perpendicular to BC. [C. U. 1952]

Theorem 8

All straight lines drawn perpendicular to a given straight at a given point are coplanar.

Let the straight lines OA, OB, OC be each perpendicular to the given straight line OP at O.

It is required to prove the OA, OB, OC *are coplanar.*

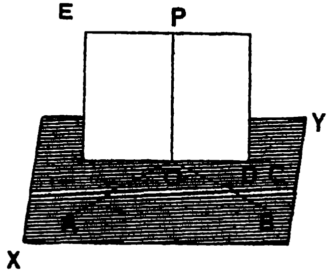


Fig. 7

Proof. Let XY be the plane through OA, OB and ED be the plane through OP, OC. Let these two planes intersect along the straight line OD.

Since OP is perpendicular to OA, OB, it is perpendicular to the plane XY and is therefore, perpendicular to the straight line OD which meets it in that plane.

But by hypothesis, OP is perpendicular to OC. Hence the three straight lines OP, OC, OD lying in the same plane are mutually perpendicular, which is absurd, unless OC and OD coincide *i.e.* OC lies in the plane XY.

In a similar manner, it may be shown that any other straight line drawn perpendicular to OP at O lies in the plane XY. *Q. E. D.*

Definitions. (i) The direction of the plumb-line through a given point in space is called the vertical line through the given point.

(ii) Any straight line drawn perpendicular to a vertical line is called a horizontal line and the plane determined by any two horizontal lines is called the horizontal plane through the point.

Any straight line drawn in the horizontal plane is perpendicular to the vertical line. Hence it follows that one and only one vertical line and an infinite number of horizontal lines may be drawn through a given point.

[C. U. 1916]

Exercises 3

1. Show that there cannot exist more than three straight lines in space which are concurrent and mutually perpendicular. [C. U. 1932, '46, '48, '51]

2. Prove that a point can be found in a plane equidistant from three points out of the plane. State the exceptional case, if any. [C. U. 1933]

Theorem 4

If two straight lines are parallel and if one of them is perpendicular to a plane, the other is also perpendicular to the same plane.

Let AB, CD be two parallel straight lines, intersecting the plane XY at B and D; let AB be perpendicular to the plane.

It is required to prove that CD is also perpendicular to the plane XY.

Join BD and draw the straight line DE in the plane XY perpendicular to BD. Join BE, AE, AD.

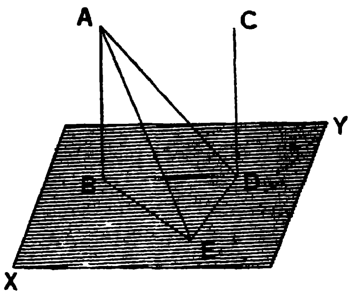


Fig. 8

Proof. Since AB is perpendicular to the plane XY, it is perpendicular to the straight lines BD and BE which lie in that plane.

$$\therefore \quad \angle ABE \text{ is a right angle}$$

$\therefore \quad AE^2 = AB^2 + BE^2 = AB^2 + BD^2 + DE^2$, ($\because$ $\angle BDE$ is a right angle) $= AD^2 + DE^2$, $\because$ $\angle ABD$ is a right angle.

$$\therefore \quad ED \text{ is perpendicular to } AD.$$

But, by construction ED is perpendicular to BD.

$\therefore$ ED is perpendicular to the plane containing AD and BD.

Now CD is a straight line in this plane, for both AD, BD lie in the plane, of the parallels AB and CD.

$\therefore$ ED is also perpendicular to CD.

Again, because AB and CD are parallel and because the $\angle ABD$ is a right angle.

$\therefore$ the $\angle CDB$ is a right angle.

So, CD being perpendicular both to DB and DE is perpendicular to the plane XY which contains them.

Q. E. D.

Exercises 4

1. If two straight lines are both perpendicular to the same plane, prove that they are parallel. (Converse of theorem 4)

[**Hints** : AB, CD are perpendicular to plane XY. With same construction and same proof, DE is perpendicular to the plane of DB and DA ; given CD is perpendicular to XY $\therefore$ perpendicular to DE $\therefore$ DB, DA, DC are coplanar and $\angle ABD$=a right angle=$\angle CDB$. $\therefore$ etc.]

2. Prove that straight lines in space which are parallel to a given straight line are parallel to one another. [C. U. 1929, '35]

3. If AB is perpendicular to the plane XY and if from B, the foot of the perpendicular, a straight line BE is drawn perpendicular to any straight line CD in the plane, then the straight line AE is also perpendicular to CD. [C. U. 1938, '40]

Theorem 5

If a straight line is perpendicular to a plane, then every plane passing through it is also perpendicular to that plane.

Let the straight line OP be perpendicular to the plane XY and let AB be any plane passing through OP.

It is required to prove that the plane AB is perpendicular to the plane XY.

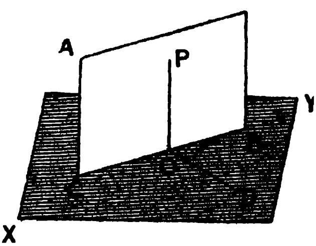


Fig. 9

Proof. Let COB be the line of section of the planes AB, XY ; also let OD be drawn in the plane XY, perpendicular to the straight line CB.

Since OP is perpendicular to the plane XY, it is perpendicular to the straight lines CB and OD, both of which lie in that plane.

$\therefore$ the $\angle POD$ is a right angle.

But, $\angle POD$ measures the dihedral angle between the planes, since OP and OD are both perpendicular to the line of section of the planes.

$\therefore$ the plane AB is perpendicular to the plane XY.

Q. E. D.

Cor. 1. If two planes are perpendicular to one another, a line drawn in one of them perpendicular to the line of section is also perpendicular to the other.

Cor. 2. If two planes are perpendicular to one another a line drawn perpendicular to one of them at any point of their line of section lies in the other plane.

Exercises 5

1. If two intersecting planes are each perpendicular to a third plane, their line of section is also perpendicular to that plane. [C. U. 1952, '58]

[Let the planes RP, RQ, intersecting along OR be both perp. to the plane XY and let them intersect the plane XY along OP and OQ respectively.

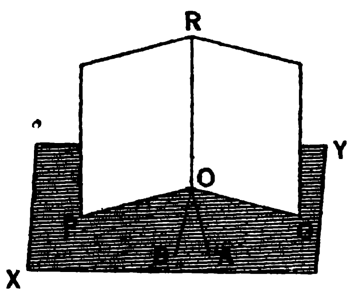


Fig. 10

Draw in the plane XY, OA perp. to OP and OB perp. to OQ. Then OA and OB are perps. to the planes RP and RQ respectively, [Cor. 1]

$\therefore$ OR is perp. to both OA and OB and therefore perp. to the plane XY.

2. Draw a plane perpendicular to a given plane and passing through a given straight line not lying in the given plane. [C. U. 1917, '19, '20]

[**Hints :** Draw a perpendicular from any point on the given straight line to the given plane. The plane through the given line and this perpendicular line is the required one.]

SOLID FIGURES

1. A finite portion of space bounded by one or more surfaces, is called a **solid figure.**

If the bounding surfaces of a solid be all planes, the solid is said to be a polyhedron. The bounding planes are called the faces and the lines of intersection of adjacent faces are called the edges of the solid.

2. A *parallelopiped* is a solid bounded by three pairs of parallel plane faces.

A *rectangular parallelopiped* is a parallelopiped whose faces are rectangular. It is also called a cuboid.

If the faces are all squares the parallelopiped is called a *cube.*

2. (a) *Expressions for the surface and volume of a rectangular parallelopiped.*

By the surface of a solid is meant the total area of the surface or surfaces of the solid and by the volume of a solid is meant the amount of the space occupied by the solid. We shall denote the surface area by S and volume by V.

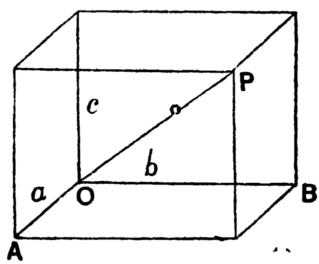


Fig. 11

If a, b, c be the lengths of the conterminous edges of a rectangular parallelopiped.

$$S = 2(ab+bc+ca). \text{ sq. units}$$
$$V = abc \text{ cubic units}$$

If the solid be a cube of side a,

$$S = 6a^2$$
$$V = a^3$$

3. A *prism* is a solid bounded by plane faces, of which two called the ends, are congruent figures in parallel planes; and the others, called the side faces are parallelograms. A

prism is said to be *right*, if the side-faces are perpendicular to the ends and so the side-faces of a right prism are rectangles.

As defined, parallelopiped is a particular case of prism and a cuboid a particular case of right prism. The perpendicular distance between the two ends is called the height of the prism.

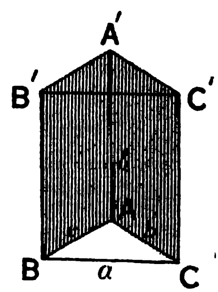


Fig. 12

3. (a) *Lateral surface of a right prism.*

= the sum of the areas of the side-faces

= the perimeter of either end $\times$ height. sq. units

Volume of a right prism = Area of either end $\times$ height. cubic units.

Thus if $2s = a + b + c + \cdots$ = the perimeter

A = the area of the either ends

h = the height of the prism

Lateral surface of the prism $= 2sh$ sq. units

$V = Ah$ cubic units

S = the total surface = the areas of the ends + the lateral surface $= 2A + 2sh$. sq. units.

4. *A pyramid* is a solid bounded by plane faces, of which one, called the base is any rectilineal figure and the others are triangles having a common vertex at a point not in the plane of the base.

The pyramid is said to be right when its base is a regular figure and the foot of the perpendicular from the vertex to the base is the central of the base.

A regular tetrahedron (or a right triangular pyramid) is a particular case of the right pyramid. It is therefore a solid bounded by four triangular faces, one of which, called the base is an equilateral triangle and the three others are congruent issosceles triangles having a common vertex.

4. (a) *Expressions for the surface and the volume of a regular tetrahedron.*

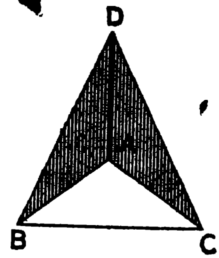


Fig. 18

Let h, the length of perpendicular from the vertex to the base = height

l, the length of perpendicular from the vertex to a side of the base = stant height

$2s$ = the perimeter of the base and
A = the area of the base.

Lateral surface of the regular tetrahedron

$= \frac{1}{2}$. stant height × perimeter of the base $= ls$ sq. units

S = the total surface = later surface + area of the base

$= ls + A$. sq. units

$V = \frac{1}{3}$ area of the base × $ht = \frac{1}{3}\, Ah$. cubic units

5. **Sphere.** *A Sphere* is a solid generated by the complete revolution of a semicircle about its bounding diameter as axis. It is evident that the distance of any point on the surface from the centre is the same. This distance is called the radius of the sphere.

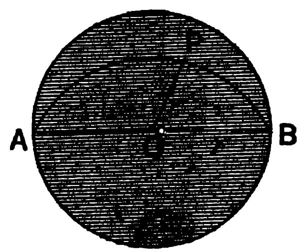


Fig. 14

5. (a) *Expressious for the surface and the volume of a sphere.*

Let R be the radius of the sphere
S, the surface $= 4\pi R^2$ sq. units
V, the volume $= \frac{4}{3}\pi R^3$ cubic units.

6. **Right circular cylinder.** *A right circular cylinder* is a solid generated by the revolution of a rectangle about one of its sides as axis. Hence the two ends, called the bases are circular and the length of straight line joining the

centres of the ends is called the height. All sections of the cylinder by planes parallel to the base are a circle and every section parallel to the axis is a rectangle.

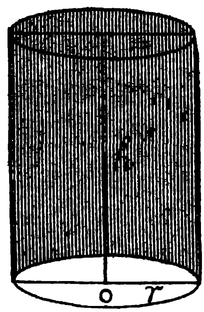


Fig. 15

6. (a) *Expressions for the surface and volume of a right circular cylinder.*

Let h = the heigt of the cylinder
r = the radius of the base.

The **lateral surface** (i. e. the curved surface)

$=$ perimeter of the base $\times$ height

$= 2\pi rh.$

S (the total surface) = area of the curved surface + the areas of the bases $= 2\pi rh + 2\pi r^2 = 2\pi r\ (h+r)$ sq. units.

V = area of the base $\times$ height $= \pi r^2 h.$

7. **Right circular cone.** *A right circular cone* is a solid generated by the revolution of a right-angled triangle about one of its sides containing the right angle as axis. Hence it may be generated by a straight line one of whose ends is fixed and the other end moves along the circumference of a circle. The fixed point is called the vertex and the straight line joining the vertex with the centre of the circular base is called the axis. The length of this axis is the height of the cone. The angle which the generating line makes with the axis is calledthe semivertical angle. By the frustum of a cone, is meant a truncated cone.

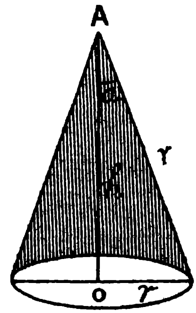


Fig. 16

7. (a) *Expressions for the surface and volume of a right circular cone.*

Let h = the height of the cone, r = the radius of the base
l = the stant height

The lateral surface (i. e. the curved surface) $= \pi rl$

$= \pi r\sqrt{h^2+r^2}$ sq. units

S = the whole surface $= \pi rl + \pi r^2$ sq. units

V = the volume of the cone $= \frac{1}{3}.\ \pi r^2 h$. cubic units.

EXAMPLES WORKED OUT

1. What is the length of one edge of a cube of which the total area of the surface is 346·56 sq. cm. ? [C. U. 1956]

Let a be the length of an edge.

$\therefore\ 6a^2 = 346{\cdot}56$ or, $a^2 = 57{\cdot}76$ $\therefore\ a = 7{\cdot}6$ c.m.

2. The length, breadth and height of a closed box are 12 in., 10 in., 8 in. in respectively and the total inner surface is 376 sq. in. ; if the walls of the box are uniformly thick, find the thickness. [C. U. 1958]

Let x'' be the thickness.

$\therefore\ 2[(12-2x)(10-2x)+(10-2x)(8-2x)+(8-2x)(12-2x)=376.$

or, $8[(6-x)(5-x)+(5-x)(4-x)+(4-x)(6-x)]=376$

or, $3x^2-30x+74=47$

or, $3x^2-30x+27=0$ or, $x^2-10x+9=0$

or, $(x-1)(x-9)=0$ $\therefore\ x=1$ or 9.

But x must be less than 4.

$\therefore$ The required thickness $=1''$.

3. The area of the whole surface of a rectangular parallelopiped is 192 sq. cm. and its volume is 144 cubic cm. If the length of a diagonal be 13 cm., find the dimensions of the solid. [C. U. 1957]

Let a, b, c be the length, breadth and height in cm.

$\therefore\ 2(ab+bc+ca)=192$; $abc=144$; $a^2+b^2+c^2=13^2$.

$\therefore\ (a+b+c)^2=(a^2+b^2+c^2)+2(ab+bc+ca)=169+192=361$

So, $a+b+c=19$.

Again, $a(b+c)+bc=96$ or $a(19-a)+\frac{144}{a}=96$

or, $a^3-19a^2+96a-144=0$ or, $(a-12)(a-4)(a-3)=0$

$\therefore$ a, the greatest side $=12$ cm.

$b=4$ c.m. and $c=3$ cm.

4. Find the volume and the lateral surface of a right prism 8 inches long standing on an issosceles triangle each of whose equal sides is 5 in. and the other side 6 in.

[C. U. 1958]

The area of triangle $=\frac{1}{2}.\ 6\sqrt{5^2-3^2}=3\times4=12$ sq in.

$\therefore$ volume $=12\times8=96$ cu. in.

the lateral surface $=(5+5+6)8=128$ sq. in.

5. Find the volume of a pyramid of which the base is a triangle whose sides are 8 cm. 15 cm. 17 cm. and the height is 12 cm. [C. U. 1957]

The base is a right angled $\triangle$ [$\because$ $17^2=8^2+15^2$], whose area $=\frac{1}{2}.\ 8\times15=60$ sq. cm.

$\therefore$ the volume $=\frac{1}{3}\times60\times12=240$ cm.

6. A solid sphere of radius 4 inches is blown into a hollow sphere of uniform thickness, radius of whose external surface is 5 inches. Find the thickness of the hollow sphere. [Given $61^{\frac{1}{3}}=3{\cdot}94$] [C. U. 1960]

If x'' be the thickness,

$\frac{4}{3}\pi\{5^3-(5-x)^3\}=$ the volume of the solid sphere

$=\frac{4}{3}\pi.4^3$

$\therefore$ $5^3-(5-x)^3=4^3$

or, $(5-x)^3=125-64=61$

$\therefore$ $5-x=61^{\frac{1}{3}}=3{\cdot}94$

$\therefore$ $x=5-3{\cdot}94=1{\cdot}06$ in.

7. A cubic inch of gold is drawn to a wire 1000 yards long ; find the diameter of the wire nearest thousandth of an inch. [C. U. 1958]

Let r'' be the radius of the wire, (which is cylindrical in shape)

$\therefore \quad \pi r^2 \times 1000 \times 12 \times 3 = 1.$

$$\text{or,} \quad 4r^2 = \frac{1}{\pi \,.\, 1000 \times 9} = \frac{1}{3{\cdot}1416 \times 100 \times 9 \times 10}$$

$$\therefore \quad 2r = \frac{1}{30 \times \sqrt{31{\cdot}416}} = \tfrac{1}{30} \times {\cdot}18 \text{ nearly} = {\cdot}006 \text{ in.}$$

8. A sphere and a right circular cylinder of the same radius have equal volumes. By what percentage does the diameter of the cylinder exceed its height ? [C. U. 1951]

Let r be the radius and h be the height of the cylinder.

$$\therefore \quad \frac{4}{3}\pi r^3 = \pi r^2 h \quad \text{or,} \quad r = \frac{3h}{4} \quad \text{or,} \quad 2r = \frac{3h}{2}.$$

Hence if h be 100, $d = \frac{3}{2} \times 100 = 150$

$\therefore$ the diameter exceeds the height by 50%.

9. A right circular cone 20 feet high has its upper part cut off by a plane passing through the middle point of its axis. If the plane of the section be at right angles to the axis and if the radius of the base of the original cone be 4 feet, find the volume of the truncated cone. [C.U. 1936]

The height of the cone cut off $= 10$ ft. and the radius of its base $= 2$ ft.

The volume of the truncated cone $=$ original volume $-$ volume of the portion cut off $= \frac{1}{3}\pi\,(4^2.\,20 - 2^2.\,10)$

$$= \frac{\pi}{3}\,(320 - 40) = \frac{280\pi}{3} \text{ cu. ft.}$$

$$= \frac{280}{3} \times \frac{22}{7} = \frac{880}{3} = 293\tfrac{1}{3} \text{ cu. ft.}$$

Exercises 6

1. Find the dimensions of a rectangular block, if the area of its total surface is 1300 sq. ft. and its length, breadth and height are in the ratio 4 : 3 : 2. [*Ans.* 20 ft., 15 ft., 10 ft.]

2. A right prism stands on triangular base whose sides are 17 cm., 10 cm., 9 cm. and the height is 10 cm. Find the volume and the whole surface. [C. U. 1940]
[*Ans.* 360 c. cm. ; 432 sq. cm.]

3. Determine the volume of a pyramid whose height is $10\sqrt{7}$ ft. and which stands on a triangle of sides 16 ft., 11 ft., 9 ft.
[*Ans.* 420. cu. ft.] [C. U. 1941]

4. Find the volume of a pyramid 16 in. high which stands on a regular base measuring 15 in. by 10 in.
[*Ans.* 800 cu. in.] [C. U. 1955]

5. Find the surface and volume of a sphere whose radius is equal to 6 inches, correct to one place of decimal.
[*Ans.* 452·4 sq. in. ; 1357·2 cu. in.] [C. U. 1954, '60]

6. Three solid spheres of glass whose radii are 1 cm., 6 cm., 8 cm. respectively are melted into a single solid sphere. Find the radius of the sphere so formed. [*Ans.* 9 cm.] [C. U. 1958]

7. The measure of the volume of a sphere is twice that of its surface. Find the radius of the sphere.
[*Ans.* 6 in.] [C. U. 1953]

8. How many solid circular cylinders each of length 8 in. and diameter 6 in. can be made out of the material of a solid sphere of radius 6 inches ? [*Ans.* 4] [C. U. 1952]

9. A cube and a sphere being of equal volume, find the ratio of the radius of the sphere to the side of the cube. [*Ans.* 31 : 50]

10. Find the lateral surface and the volume of a right circular cone 15 ft. high, the radius of whose base is 8 ft.
[*Ans.* $427\frac{3}{7}$ sq. ft. ; $1005\frac{5}{7}$ cu. ft.]

11. Show how to draw a plane parallel to the base of right circular cone so that it divides the cone into two parts of (*i*) equal surface ; (*ii*) equal volumes.
[*Ans.* The plane divides the axis in the ratio (*i*) $\sqrt{2}-1 : 1$ and (*ii*) $\sqrt[3]{2}-1 : 1$]

12. The stant height of a conical tomb is 35 ft. If its diameter is 56 ft. find the cost of constructing it at Re. 1. 4 as. 3 p. per cubic yard and also the cost of white washing its curved surfaces at 5 as. per 100 sq. ft. [*Ans.* Rs. 808 – 8 ; Rs. 9 – 10]

ANSWERS

CO-ORDINATE GEOMETRY

Exercises 1

1. (i) $3\sqrt{10}$ (ii) $\sqrt{a^2+b^2}$ (iii) $a(t_1 - t_2)\sqrt{(t_1+t_2)^2 \cdot}$

2. $3 \pm \sqrt{5}$ 8. $(\frac{10}{7}, \frac{33}{7})$

10. (i) 5 (ii) a^2 (iii) $a^2(t_1 - t_2)(t_2 - t_3)$

11. (i) $20\frac{1}{2}$ (ii) 96 14. $(-\frac{1}{14}, \frac{39}{14})$

Exercises 3 (A)

1. (a) $y - x = 1$ (b) $ax - by = ab$

   (c) $y(t_1 + t_2) - 2x = 2at_1t_2$ (d) $y + x = 2$

2. $x - y - 5 = 0$ 3. $x - y\sqrt{3} - 2\sqrt{3} = 0$

4. $5y - 8x = 60$ 5. $20y - 9x = 96$

6. (i) $x + 3y + 7 = 0$, $y - 3x = 1$, $y + 7x = 11$

   (ii) $2x - 3y = 4$, $y - 3x = 1$, $x + 2y = 2$

7. $y(a' - a) - x(b' - b) = a'b - ab'$

   $y(a' - a) + x(b' - b) = a'b' - ab$

9. $7x + 3y = 13$ 10. $x - 2y + 8 = 0$, -8 and 4

11. (i) $x \cos 225^\circ + y \sin 225^\circ = \sqrt{2}$

    Length of perpendicular $= \sqrt{2}$

    (ii) $x \cos 240^\circ + y \sin 240^\circ = 7$,, ,, $= 7$

12. $x - y + 1 = 0$

Exercises 3 (B)

1. $3x + 4y + 8 = 0$

2. (i) $\tan^{-1} \frac{4}{3}$ (ii) $\tan^{-1} \frac{23}{10}$ (iii) 90°

3. (i) $7x + 5y = 7$ (ii) $2a(y - y') + y'(x - x') = 0$

4. $(x - x_1)(x_2 - x_3) + (y - y_1)(y_2 - y_3) = 0$

5. $2x(a - a') + 2y(b - b') = a^2 - a'^2 + b'^2 - b^2$

6. (i) $\frac{a}{m_1 m_2}$, $a\left(\frac{1}{m_1}+\frac{1}{m_2}\right)$ (ii) $\left(\frac{ab}{a+b}, \frac{ab}{a+b}\right)$

8. (i) $(\frac{1}{2}, \frac{1}{2})$ (ii) $\left[\frac{b^2\alpha - ab\beta - ca}{a^2+b^2}, a^2\beta - ab\alpha - bc\right]$

9. $43x - 29y = 71$ 10. $x+y+2=0$

11. $3x+4y=5a$ 12. $23x+23y=11$

14. $(2, 1)$ 15. $3x - y - 7 = 0$, $\frac{49}{8}$

16. $x=3$; $y=4$

Exercises 3 (C)

1. (a) (i) $x - 2y + 1 = 0$, (ii) $2x + y = 3$

(b) (i) $x(2\sqrt{2} - 3) + y(\sqrt{2} - 1) = 4\sqrt{2} - 5$

(ii) $x(2\sqrt{2} + 3) + y(\sqrt{2} + 1) = 4\sqrt{2} + 5$

2. $4\frac{2}{5}$ 5. $\frac{c-d}{\sqrt{1+m^2}}$ 6. $\left\{\frac{a}{b}(b \pm \sqrt{a^2+b^2}), 0\right\}$

7. $\sqrt{h^2+k^2}$, 8. $4\frac{1}{2}$ 12. $2x - 6y = 1$ or, $6x + 2y - 5 = 0$

Exercises 4

1. (a) Centre $(2, 4)$ radius $= \sqrt{61}$

(b) " $(2, 2)$ " $= \sqrt{2}$

(c) " $(\frac{1}{5}, \frac{1}{5})$ " $= \sqrt{\frac{13}{10}}$

(d) " $(g, -f)$ " $= \sqrt{g^2+f^2}$

(e) " $(-3, 4)$ " $= 7$

2. (a) $3x^2+3y^2-29x-19y+56=0$

(b) $b(x^2+y^2)-(a^2+b^2)x+(a-b)(a^2+b^2)=0$

(c) $4x^2+4y^2-142x+47y+138=0$

(d) $x^2+y^2-22x-4y+25=0$

4. $x^2+y^2-3x-4y=0$ 5. $x^2+y^2 \pm 2ax \pm 2by + a^2 = 0$

6. $b(x^2+y^2) = x(b^2+c^2)$ 7. $(x-2)(x-5)+(y-3)(y-6)=0$

9. $x^2+y^2-2x-2y+1=0$ 11. $x^2+y^2 \pm 2\sqrt{21}y - 4 = 0$

12. $x^2+y^2 \pm 6\sqrt{2}y - 6x + 9 = 0$ 13. $x^2+y^2-hx-ky=0$

14. $x^2+y^2+8x-10y+16=0$

15. $x^2+y^2+8x-10y-89=0$

16. $5x-12y=152, \quad 24x+10y+1$

17. $x+2y=\pm 2\sqrt{5}$

18. $4x+3y+19=0$ and $4x+3y-31=0$

21. $\left(-\frac{c}{\sqrt{2}}, \frac{c}{\sqrt{2}}\right)$ 22. $Aa+Bb+c=\pm c\sqrt{A^2+B^2}$

23. $x+5=0 \quad y-2=0$

25. $c(l^2+m^2)+n^2-(fl-gm)^2-2fmn-2gnl=0$

$g^l+fm-n=0$

Exercises 5

1. $16x^2+y^2+8xy-74x-78y+212=0$

2. $(ax-by)^2-2a^3x-2b^3y+a^4+a^2b^2+b^4=0$

3. (i) $(-1, 2), y=2, 4, (0, 2)$ (ii) $(1, 2), y=2, 4, (0, 2)$

4. $4y=3x+12, 4y+4x=34$

5. $y-x=3, y+x=9, \quad x+y+3=0, x-y=9$

6. $\left(\frac{a}{3}, \frac{2a}{\sqrt{3}}\right)$ 8. $(3a, 2\sqrt{3}a)$ $\left(\frac{a}{3}, \frac{-2\sqrt{3}}{3}a\right)$

9. $\frac{4}{3}, (\frac{1}{3}, 0)$ 10 $x^2+4y^2-4xy+4x+2y-1=0$

11. $(0, 1)$ 12. $\left(\frac{a}{3}, \frac{-2a}{\sqrt{3}}\right)$

14. $16x-12y+3=0$ $(\frac{3}{10}, \frac{1}{2})$

15. (i) $my=x+am^2, \quad y+mx=2am+am^3$

Exercises 6

1. (a) foci $(\pm 4, 0)$, $c=4$, $4x=\pm 25$,

(b) $e=\frac{1}{\sqrt{5}}$ foci $\left(\pm \frac{1}{2\sqrt{3}}, 0\right)$

2. $20x^2+36y^2=405$ 3. $4x^2+9y^2+16x-5y+61=0$

4. $\frac{\sqrt{3}}{2}$ 5. $8x^2+9y^2=1152$

6. $7x^2+2xy+7y^2+10x-16y+7=0$

7. $x+3y=5$, $9x-3y-5=0$

8. $\pm\sqrt{7}x \mp 4y=16$, $\pm 4x \mp y\sqrt{7}=\frac{7}{4}\sqrt{7}$

10 $y=3x\pm\frac{1}{2}\sqrt{\frac{195}{2}}$ 11. $2x+3y-6=0$, $9x-6y-14=0$

12. $y=2x\pm\sqrt{\frac{7}{5}}$ 13. $\pm\frac{\pi}{4}$, $\pm\frac{3\pi}{4}$

14. $25x^2-144y^2=90°$ 15. $65x^2-36y^2=441$

16. $(6, 4)$ $(\pm 13, 0)$ $2\frac{2}{3}$ 17. $24y-30x=\pm\sqrt{161}$

18. $x+2y=\pm 2\sqrt{2}$ 19. $\frac{13}{12}$ $(\pm 13, 0)$

20. $5x^2-4y^2-20x+24y-36=0$

22. $x-3y+2=0$ $(-1, \frac{1}{3})$, $x-3y-2=0$ $(1, -\frac{1}{3})$

CALCUTTA UNIVERSITY QUESTIONS

1958

[*In the following questions, assume the axes as rectangular.*]

1. (a) Prove that the lines joining the mid-points of opposite sides of any quadrilateral bisect each other.

(b) Show that the points $(a, b+c)$, $(b, c+a)$, $(c, a+b)$ are collinear.

2. (a) Find the equation of the line bisecting the join of $(2a, 2b)$, $(2c, 2d)$ at right angles.

(b) If p_1 and p_2 be the perpendiculars from the origin upon the straight lines

$$x \sin\theta + y \cos\theta = \tfrac{1}{2} a \sin 2\theta$$

$$\text{and} \quad x \cos\theta - y \sin\theta = a \cos 2\theta,$$

prove that $4p_1^2 + p_2^2 = a^2$.

3. (a) Find the equation of the circle described on the diameter whose end points have the co-ordinates (x_1, y_1) and (x_2, y_2).

(b) Find the equations of the tangents to the circle $x^2 + y^2 = 25$ parallel to the line $3x + 4y = 0$.

4. (a) Obtain the equation of the parabola whose focus is at the point $(3, -\tfrac{1}{2})$ and whose directrix is the straight line $2x - y + 3 = 0$.

Prove that the straight line $\dfrac{ax}{3} + \dfrac{by}{4} = c$ will be a normal to the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, if $5c = a^2c^2$.

5. (a) The length, breadth and height of a closed box are 12 in. 10 in., and 8 in. respectively and the total inner surface is 376 sq. in. ; if the walls of the box are uniformly thick, find the thickness.

(b) A cubic inch of gold is drawn into a wire 1000 yards long ; find the diameter of the wire to the nearest thousandth of an inch. $(\pi = 3{\cdot}1416)$

1959

1. (*a*) Find the area of the triangle, the co-ordinates of whose angular points are (x_1, y_1), (x_2, y_2) and (x_3, y_3).

(*b*) If the points (a, b), (a', b'), $(a-a', b-b')$ are collinear, show that $ab' = a'b$.

2. (*a*) Find the length of the perpendicular from a given point upon a given straight line.

(*b*) Prove that the sum of the squares of the perpendiculars from the origin upon the lines.

x sec $a + y$ cosec $a = 2a$, x cos $a + y$ sin $a = a$ cos $2a$ is independent of a.

3. (*a*) Find the equation to the circle which passes through the points (0, 0), (6, 0) and (0, 8).

(*b*) Find the equation to the chord of the circle $x^2 + y^2 = 81$, which is bisected at P $(-2, 3)$.

4. (*a*) Find the equation of the tangent to the parabola $y^2 = 4ax$ at any point (x', y') upon it.

(*b*) Find the point of the parabola $y^2 = 8x$, at which the normal is inclined at 60° to the axis of the parabola.

5. (*a*) Write down (without proof) the expressions for the volumes of (*i*) a right prism, (*ii*) a triangular pyramid and (*iii*) a right cone.

(*b*) Find the volume of the pyramid of which the base is a triangle whose sides are 8 cm., 15 cm., and 17 cm , and the height is 12 cm.

1960

1. (*a*) Find the co-ordinates of the point which divides the traight line joining (x_1, y_1) and (x_2, y_2) *internally* in the ratio $n : n$.

(*b*) The side BC of a triangle ABC is divided internally at D in the ratio $n : m$ and the line AD is divided internally at F in the ratio $m+n : l$. If the co-ordinates of the angular

points A, B, C be (x_1, y_1), (x_2, y_2) and (x_3, y_3) respectively, find the co-ordinates of G.

2. (a) Find the angle between the straight lines

$$ax + by + c = 0.$$
$$\text{and } a'x + b'y + c' = 0$$

(b) Find the equation of the straight line passing through the intersection of $x + 2y = 0$ and $y + 4x + 7 = 0$ and perpendicular to the straight line $3x - y = 0$.

3. (a) Find the equation of the tangnet at any point (x', y') of the circle whose equation is $x^2 + y^2 + 2gx + 2fy + c = 0$.

(b) Show that $y = mx$ is a tangent to the circle

$$x^2 + y^2 + 2gx + 2fy + c = 0 \text{ if } (g + mf)^2 = c(1 + m^2).$$

4. (a) Find the equation of the normal at the point (x_1, y_1) to the parabola $y^2 = 4ax$.

(b) Prove that the line $2x + 4y = 9$ is a normal to the parabola $y^2 = 8x$, and find the co-ordinates of the point of the parabola at which it is the normal.

5. (a) Find the surface and volume of a sphere whose radius is equal to 6 inches, correct to one place of decimal. ($\pi = 3{\cdot}1416$).

(b) A solid sphere of radius 4 inches is blown into a hollow sphere of uniform thickness, radius of whose external surface is 5 inches. Find the thickness of the hollow sphere, given $61^{\frac{1}{3}} = 3{\cdot}94$.